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A Method for Determining the Individual Accuracy of Astrometric Catalogues

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Abstract. We present a method for determining empirically the actual accuracy (external errors) of proper motions and positions given in astrometric catalogues. The method is based on a comparison of a sufficient number of independent proper motions and positions. The method can be applied, for example, to a comparison of the basic FK5 and the future Hipparcos catalogue with the aim to determine the external errors of the proper motions in both catalogues separately. A few variations of the same basic method are presented: The t_{min} method, the μ_0 method, and the K method. The expected statistical uncertainty of the derived external errors is discussed, especially as a function of the number of stars used. Numerically simulated catalogues confirm the validity of the results produced by the proposed method. The most disturbing effect may be undetected astrometric binaries.

Key words: astrometry: proper motions – astrometry: positions – astrometry: external errors

1. Introduction

We describe here a method to determine the actual (external) accuracy of proper motions given in two independent astrometric catalogues. The method has been developed during the evaluation of the results of the Hipparcos data reduction in which the reliability of the Hipparcos proper motions is tested by comparing them with the very accurate ground-based proper motions of the basic FK5. The method is, however, quite general and can be applied both to available ground-based catalogues as well as to future space-born results. In this paper, we present the basic method, discuss the expected accuracy of the method, and give the results of a numerical test of the method based on simulated data.

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2. The basic problem

Let us assume that we have two independently obtained astrometric catalogues, such as the expected Hipparcos catalogue and the basic FK5 (Fricke et al. 1988). After eliminating systematic errors of various kinds (see Sections 3), we determine for each star the difference $\Delta\mu_{21} = \mu_2 - \mu_1$ of the proper motions μ_1 and μ_2, given in the Catalogue 1 and 2, respectively (μ denotes one of the components of the proper motion, such as μ_δ). We form then the root-mean-squared (rms) value of $\Delta\mu_{21}$, which we call D_{21}, averaged over stars common to both catalogues. The expectation value of D^2_{21} is $\epsilon^2_{\mu,1} + \epsilon^2_{\mu,2}$, where $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ are the actual (external) mean errors of μ_1 and μ_2. Is it also possible to determine $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ *separately* for each catalogue, instead of deriving only the combination $\epsilon^2_{\mu,1} + \epsilon^2_{\mu,2}$? In contrast to a negative expectation which I had first, it turned out that the solution of the problem is possible, if the two catalogues contain, beside the proper motions, also positions at two different epochs of time. This condition is, for example, nicely fulfilled for the FK5 and the Hipparcos catalogue.

In Sections 4 to 6, we shall present three solutions to the problem of determining $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ separately: the t_{min} method, the μ_0 method, and the K method. In principle, all the three methods are identical, and they produce the same answers. At first glance, they look, however, quite different, and they illustrate thereby different aspects of the same basic idea.

3. The basic assumptions and the preparation of the catalogue data

We consider two astrometric catalogues (with the indices 1 and 2) which contain completely independent data, i.e. the results presented in the two catalogues should be derived from two completely distinct sets of observations, such as it is the case for the FK5 and the Hipparcos catalogue. Both catalogues should give, for a common set of stars, proper motions and positions for each star in two coordinates (e.g. in equatorial coordinates α and δ). Furthermore, both catalogues should give for each star and each component a central epoch, T_1 and T_2, at which the position is not correlated with the proper motion. The central epoch T is often explicitely given in the catalogue, such as in the FK5. In the Hipparcos catalogue, all the five astrometric parameters of a star (two components of the position, two components of the proper motion, and the parallax) are correlated with each other. From the known correlation coefficients, it is possible to derive for each star and each component (α or δ) separately a central epoch T at which the position and the proper motion for this component are uncorrelated. In general, however, the remaining correlation coefficients for this epoch T will not vanish. This fact is probably not a severe limitation for real applications. Firstly, the Hipparcos positions have negligible errors compared to ground-based positions over the whole period of the Hipparcos observations. Secondly, the possible range of central epochs for Hipparcos stars is in many cases negligibly small with respect to the epoch difference $T_2 - T_1$ (e.g. about 40 years for Hipparcos and FK5). Hence even the use of a common (mean) central epoch T for all the Hipparcos stars and for both components would not severely affect the results.

Before comparing the positions and proper motions of each star in both catalogues, the systematic differences between the catalogues must be removed from the positions and proper motions. In other words, one catalogue has to be reduced to the system of the other catalogue. The reason for systematic differences between the two catalogues may be local distortions of the reference frame (such as of the FK5 system) or different orientations of the reference frames (e.g. of the FK5 system with respect to the Hipparcos system). In our example, the FK5 should be reduced to the Hipparcos system. An arbitrary rotation of the Hipparcos system of positions and proper motions with respect to an extragalactic reference frame would not affect our method.

It is beyond the scope of this paper to discuss adequate methods for the removal of the systematic differences between the two catalogues. It may be sufficient to mention that we envisage methods (Bien et al. 1978, Schwan 1983) which have been used for the construction of the FK5 and which are also applicable for deriving systematic differences between the proper motion systems of two catalogues.

After having removed the systematic differences between the two catalogues, we are now aiming at determining the 'random' errors of the proper motions and positions given in the two catalogues. In the classical astrometric literature, these 'random' errors are called 'individual' errors (in contrast to the 'systematic' errors which, by definition, affect an ensemble of stars). We are now forming for each star the differences between the two catalogues for each component of the proper motion,

$$\Delta\mu_{21} = \mu_2 - \mu_1 \quad , \tag{1}$$

and for each component of position at an epoch t,

$$\Delta x_{21}(t) = x_2(t) - x_1(t) \quad . \tag{2}$$

For writing economy, we suppress here the index i of the star and the index for the component. For example, at length, we should read for x_1 the quantity $\alpha_{1,i} \cos\delta_{1,i}$ or $\delta_{1,i}$, for μ_2 the quantity $\mu_{2,i,\alpha} \cos\delta_{2,i}$ or $\mu_{2,i,\delta}$. We emphasize again that the quantities μ and x, and hence $\Delta\mu$ and Δx, should not be taken directly from the catalogues, but should be first corrected for the systematic differences between the two catalogues.

Most catalogues give the position (and proper motion) of a star at some standard epoch (e.g. at J2000.0 in the FK5). The basic physical assumption in transforming positions from one epoch t' to another epoch t is that every star moves linearly in time on a straight line. This means, to a first approximation, that the spherical coordinates change linearly with time, e.g.

$$x_1(t) = x_1(t') + \mu_1(t - t') \quad . \tag{3}$$

We shall neglect here spherical effects and effects of the changing perspective (called 'foreshortening' in classical astrometry) for simplicity. In real applications, both effects can be easily accounted for, especially as far as the comparison between two catalogues is concerned. Nevertheless, the assumption in Eq.(3) is the most severe and limiting one for actual applications, since it neglects the orbital motions of undetected astrometric binaries (see Section 12) among the stars included into the common set of catalogue stars. Predictions on the effect of astrometric binaries on Hipparcos proper motions have been made by Lindegren (1979) and Soederhjelm (1985).

In order to formulate our basic statistical assumptions we have now to introduce the concept of the 'true' values of μ and $x(t)$, which we denote by μ_w and $x_w(t)$. The true values are, of course, unknown in reality. The differences between the catalogue values and the true values, in the common reference system, are given by

$$\begin{aligned} \Delta\mu_{1w} &= \mu_1 - \mu_w \quad , \\ \Delta\mu_{2w} &= \mu_2 - \mu_w \quad , \\ \Delta x_{1w}(t) &= x_1(t) - x_w(t) \quad , \\ \Delta x_{2w}(t) &= x_2(t) - x_w(t) \quad . \end{aligned} \tag{4}$$

It is not necessary that the chosen reference system is in any sense a 'true' reference frame. The only requirement is that all the data, including μ_w and $x_w(t)$, are transformed into the *same* reference frame, say that of Catalogue 2 for example.

The basic statistical assumption of our method is now the following: The differences $\Delta\mu_{1w}$, $\Delta\mu_{2w}$, $\Delta x_{1w}(t)$, and $\Delta x_{2w}(t)$ of the stars in the two catalogues are distributed symmetrical around zero and have a finite dispersion around zero. This assumption includes the statement that the mean values of all these differences are zero within the statistical noise. The requirement of the finite dispersions ensures that we can characterize the distribution functions of the differences by their second-order moments (taken around zero), i.e. by the rms average of the differences which we denote by $\epsilon_{\mu,1}, \epsilon_{\mu,2}$, $\epsilon_{x,1}(t)$, and $\epsilon_{x,2}(t)$:

$$\begin{aligned} \epsilon_{\mu,1} &= <(\Delta\mu_{1w})^2>^{1/2} , \\ \epsilon_{\mu,2} &= <(\Delta\mu_{2w})^2>^{1/2} , \\ \epsilon_{x,1}(t) &= <(\Delta x_{1w}(t))^2>^{1/2} , \\ \epsilon_{x,2}(t) &= <(\Delta x_{2w}(t))^2>^{1/2} \quad , \end{aligned} \tag{5}$$

where the operator $< >$ denotes the mean of the quantities q_i $(i = 1, \ldots, n)$ taken over the n stars which are common to the two catalogues:

$$<q_i> = \frac{1}{n}\sum_{i=1}^{n} q_i \quad . \tag{6}$$

The quantities ϵ are the actual (external) mean errors of the corresponding quantities, which we wish to determine by comparing the two catalogues. In most cases, the catalogues give the expected (internal) errors σ_i of the quantities, e.g. of μ_1 and $x_1(T_1)$, for each star. These internal mean errors σ_i are estimated usually from the internal agreement among the measurements which lead e.g. to the values of μ_1 and $x_1(T_1)$ given in the Catalogue 1. For comparison with ϵ, we form an rms average of these individual values over the n stars:

$$\sigma = <\sigma_i^2>^{1/2} \quad , \tag{7}$$

i.e. for $\sigma_{\mu,1}$, $\sigma_{\mu,2}$, $\sigma_{x,1}(T_1)$, $\sigma_{x,2}(T_2)$. Ideally, the external error should be equal to the internal one, $\epsilon = \sigma$. Our method provides a mean to test this.

The statistical assumption stated above should be nicely fulfilled in most applications. We do not require that the distribution functions of the differences $\Delta\mu_{1w}$, $\Delta\mu_{2w}$, Δx_{1w}, Δx_{2w} are gaussian. The most dangereous deviation from our assumption would probably be a skewness in the distribution functions. In real applications, our statistical assumption can be partially checked empirically by investigating the distribution functions of the available quantities $\Delta\mu_{21}$ and of Δx_{21}. These distribution functions should be symmetrical around zero, and the mean values, $<\Delta\mu_{21}>$ and $<\Delta x_{21}(t)>$, should be zero within the statistical noise. Of course, these conditions are only necessary but not sufficient for the validity of our statistical assumptions.

4. The t_{min} method

The t_{min} method uses the fact that the accuracy of the positions predicted by each of the two catalogues k ($k = 1, 2$) is a strong function of time t:

$$\sigma_{x,k}^2(t) = \sigma_{x,k}^2(T_k) + \sigma_{\mu,k}^2(t - T_k)^2 , \tag{8}$$

Fig. 1. Schematic illustration of the t_{min} method by an artificial example. The full curve represents the rms average (over the stars common to both catalogues) of the differences of the positions of each star in the two catalogues as a function of epoch t: $d_{21}(t) =<(x_2(t) - x_1(t))^2>^{1/2}$. The dashed lines are the mean errors $\sigma_x(t)$ $(= \epsilon_x(t)$ in this example) of a position in a catalogue as a function of epoch. The input data for the two simulated catalogues are the same as those of Table 1 in Section 10.

where T_k is the central epoch in catalogue k. For simplicity, we shall assume here that either T_k is the same for all the stars in catalogue k, or that we can use a mean value $T_k = <T_{k,i}>$ instead of the individual central epoch $T_{k,i}$ of each star. The other methods (Sections 5 and 6) allow more easily to take individual values $T_{k,i}$ into account.

In Fig.1, we illustrate the behaviour of $\sigma_{x,1}(t)$ and $\sigma_{x,2}(t)$ for two catalogues which correspond roughly to the basic FK5 and the expected Hipparcos catalogue for bright stars. The minimum of $\sigma_{x,k}(t)$ is reached at the central epoch T_k. For epochs t far away from T_k, $\sigma_{x,k}(t)$ grows linearly with time.

We are now investigating the behaviour of $\Delta x_{21}(t)$, defined by Eq.(2) and calculated from the following Eq.(9):

$$\begin{aligned} \Delta x_{21}(t) &= x_2(t) - x_1(t) \\ &= x_2(T_2) + \mu_2(t - T_2) \\ &\quad -x_1(T_1) - \mu_1(t - T_1) \quad . \end{aligned} \tag{9}$$

We form the rms average of $\Delta x_{21}(t)$ over the n stars:

$$d_{21}(t) = <(\Delta x_{21}(t))^2>^{1/2} \quad . \tag{10}$$

The expectation value of d_{21}^2(t) is given by

$$d_{21}^2(t) = <(x_2(T_2) - x_w(T_2) - (x_1(T_1) - x_w(T_1))$$

$$
\begin{aligned}
& + (\mu_2 - \mu_w)(t - T_2) - (\mu_1 - \mu_w)(t - T_1))^2 > \\
= & < (\Delta x_{2w}(T_2) - \Delta x_{1w}(T_1) \\
& + \Delta\mu_{2w}(t - T_2) - \Delta\mu_{1w}(t - T_1))^2 > \\
= & < (\Delta x_{1w}(T_1))^2 + (\Delta x_{2w}(T_2))^2 \\
& + (\Delta\mu_{1w})^2 (t - T_1)^2 + (\Delta\mu_{2w})^2 (t - T_2)^2 > \\
= & \ \epsilon_{x,1}^2(T_1) + \epsilon_{x,2}^2(T_2) \\
& + \epsilon_{\mu,1}^2 (t - T_1)^2 + \epsilon_{\mu,2}^2 (t - T_2)^2 \quad , \qquad (11)
\end{aligned}
$$

since the mixed terms such as $< \Delta x_{1w}(T_1)\Delta\mu_{1w} >$ or $< \Delta\mu_{1w}\Delta\mu_{2w} >$ are vanishing because of the independence of the two catalogues and the non-correlation of μ_k and x_k at the central epoch T_k. We can rearrange the last part of Eq.(11) into a form similar to Eq.(8):

$$
\begin{aligned}
d_{21}^2(t) = & \ \epsilon_{x,1}^2(T_1) + \epsilon_{x,2}^2(T_2) + \frac{\epsilon_{\mu,1}^2\epsilon_{\mu,2}^2}{\epsilon_{\mu,1}^2 + \epsilon_{\mu,2}^2}(T_2 - T_1)^2 \\
& + (\epsilon_{\mu,1}^2 + \epsilon_{\mu,2}^2)(t - t_{min})^2 \quad , \qquad (12)
\end{aligned}
$$

where t_{min} is given by

$$
t_{min} = \frac{\epsilon_{\mu,1}^2 T_1 + \epsilon_{\mu,2}^2 T_2}{\epsilon_{\mu,1}^2 + \epsilon_{\mu,2}^2} \quad . \qquad (13)
$$

In Fig.1, we have plotted $d_{21}(t)$ for the case that the internal and external errors would agree ($\epsilon = \sigma$). The rms difference of the positions derived from the two catalogues according to Eqs.(9) and (10), should show a minimum at t_{min} according to Eq.(13). If one of the catalogues, say Catalogue 2, were completely free of errors, $d_{21}(t)$ would follow exactly the curve $\sigma_{x,1}(t)$ of the other Catalogue 1, and t_{min} would be equal to the central epoch T_1. If both catalogues had the same accuracy of proper motions, $\epsilon_{\mu,1} = \epsilon_{\mu,2}$, then t_{min} would be the mean of T_1 and T_2. This and Eq.(13) show clearly that the position of t_{min} relative to T_1 and T_2, which can be determined empirically from the run of $d_{21}(t)$, gives us the possibility to determine $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ separately, if T_1 and T_2 are significantly different from each other.

In practice, we should proceed as follows: We calculate the individual values $\Delta x_{21}(t)$ for a few instants of time t, and form the rms average $d_{21}(t)$. The values of $d_{21}^2(t)$ are fitted by a parabola:

$$
d_{21}^2(t) = a_0 + a_1 t + a_2 t^2 \quad . \qquad (14)
$$

From a_1 and a_2, we determine t_{min}:

$$
t_{min} = -a_1/(2a_2) \quad . \qquad (15)
$$

Equation (12) shows that

$$
\epsilon_{\mu,1}^2 + \epsilon_{\mu,2}^2 = a_2 \quad . \qquad (16)
$$

This quantity can also be derived directly from the rms average of the proper motion differences:

$$
D_{21}^2 = < (\Delta\mu_{21})^2 > = \epsilon_{\mu,1}^2 + \epsilon_{\mu,2}^2 \quad . \qquad (17)
$$

From t_{min}, we can now derive $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$:

$$
\begin{aligned}
\epsilon_{\mu,1}^2 &= D_{21}^2 \, \frac{T_2 - t_{min}}{T_2 - T_1} \quad , \\
\epsilon_{\mu,2}^2 &= D_{21}^2 \, \frac{t_{min} - T_1}{T_2 - T_1} \quad . \qquad (18)
\end{aligned}
$$

How many instants of time are required for fitting Eq.(14) to $d_{12}^2(t)$? Both in theory and reality, three (well-chosen) instants of time are sufficient. Since Eq.(3) is used for transforming positions from one epoch to another, each individual difference $\Delta x_{21}(t)$ follows the law

$$
(\Delta x_{21}(t))^2 = b_0 + b_1 t + b_2 t^2 \quad . \qquad (19)
$$

Hence the mean of all individual Equations (19) follows strictly Eq.(14), with $a_0 = < b_0 >$, $a_1 = < b_1 >$, $a_2 = < b_2 >$. This property is true in reality, however, only if we determine the systematic differences between the catalogues (Section 3) in the following way: We determine the systematic differences in position at one epoch, e.g. at T_1, and separately the systematic differences in proper motion. The systematic differences in position at any other epoch t are then derived from an equation similar to Eq.(3) by a linear transformation. If we determine the systematic differences in position for various times t directly (without using the systematic differences in proper motion), this would violate the strict validity of Eq.(19), because of some noise in the determination of such systematic differences. In this case, a large number of fitting points in time are helpful for determining the coefficients a_0, a_1, a_2 of Eq.(14).

Equation (12) gives us also a disappointing message: a separate determination of the positional errors $\epsilon_{x,1}(T_1)$ and $\epsilon_{x,2}(T_2)$ of the two catalogues is not possible, since we can determine only the combination $\epsilon_{x,1}^2(T_1) + \epsilon_{x,2}^2(T_2)$. In some cases, however, we may either have reasons to believe that in one of the catalogues (say 2) the internal error of position is equal to the external one ($\epsilon_{x,2}(T_2) = \sigma_{x,2}(T_2)$) or that $\epsilon_{x,1}(T_1) \gg \epsilon_{x,2}(T_2)$. In both cases, we can then determine $\epsilon_{x,1}(T_1)$ using Eqs.(12),(13),(14), and (18). The combination $\epsilon_{x,1}^2(T_1) + \epsilon_{x,2}^2(T_2)$ can be also determined in the classical way from:

$$
\begin{aligned}
& \epsilon_{x,1}^2(T_1) + \epsilon_{x,2}^2(T_2) \\
& \quad = d_{21}^2(T_1) - \epsilon_{\mu,2}^2(T_2 - T_1)^2 \\
& \quad = d_{21}^2(T_2) - \epsilon_{\mu,1}^2(T_2 - T_1)^2 \quad , \qquad (20)
\end{aligned}
$$

after we have obtained $\epsilon_{\mu,1}$ or $\epsilon_{\mu,2}$ empirically by one of our methods.

We would like to remark that Eq.(18) provides the simplest way to study the effect of possible errors of T_1 and T_2 on $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$. Such errors in T could occur if the internal weights of the individual observations on which a catalogue is based were wrong. In general, a least-squares solution of such observations produces at the same time

T, $x(T)$, and μ, and an error in the weights of the observations would affect all the three quantities. If, however, a catalogue is based on two observations only, then $x(t)$ and μ are independent of the weights, while T and hence $x(T)$ would depend on the weights. In the case of many observations, it may also be a good first approximation to assume that $x(t)$ and μ are also not severely affected by wrong weights, in contrast to T. Then $d_{21}(t)$, given by Eq.(10), does not depend on T_1 or T_2. Hence the empirical values of t_{min} and of a_2 or $D_{21}^2 = \epsilon_{\mu,1}^2 + \epsilon_{\mu,2}^2$ are also not affected by changes in T_1 and T_2. It is then possible to calculate $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ as a function of T_1 or T_2, using Eqs.(18). For example, we could assume that Catalogue 2 is 'perfect', i.e. that T_2 is correct and $\epsilon_{\mu,2} = \sigma_{\mu,2}$, and solve Eq.(18), giving $\epsilon_{\mu,1}$, for a resulting value of T_1.

5. The μ_0 method

The fact that the two catalogues give positions at two different epochs allow us to determine, beside the two proper motions μ_1 and μ_2, a third proper motion μ_0 for each star (Wielen 1988):

$$\mu_0 = \frac{x_2(T_2) - x_1(T_1)}{T_2 - T_1} \quad . \tag{21}$$

Here, we can easily make use of individual central epochs T_1 and T_2 for each star. Since the quantities $x_k(T_k)$ and μ_k ($k = 1,2$) are uncorrelated at the epoch T_k, the new proper motion μ_0 is independent of and not correlated with both μ_1 and μ_2.

Using μ_0, μ_1, μ_2 of each star, we can determine three differences per star, $\mu_2 - \mu_1, \mu_1 - \mu_0$, and $\mu_2 - \mu_0$. We then take the rms average of these differences over the n stars common to both catalogues:

$$\begin{aligned} D_{21} &= < (\mu_2 - \mu_1)^2 >^{1/2} \quad , \\ D_{10} &= < (\mu_1 - \mu_0)^2 >^{1/2} \quad , \\ D_{20} &= < (\mu_2 - \mu_0)^2 >^{1/2} \quad . \end{aligned} \tag{22}$$

The expectation value of D_{21}^2 is given by

$$\begin{aligned} D_{21}^2 &= < ((\mu_2 - \mu_w) - (\mu_1 - \mu_w))^2 > \\ &= < (\Delta\mu_{2w} - \Delta\mu_{1w})^2 > \\ &= < (\Delta\mu_{1w})^2 > + < (\Delta\mu_{2w})^2 > \\ &= \epsilon_{\mu,1}^2 + \epsilon_{\mu,2}^2 \quad , \end{aligned} \tag{23}$$

since the mixed term $< \Delta\mu_{1w}\Delta\mu_{2w} >$ vanishes because of the independence of μ_1 and μ_2. In a similar way, we obtain

$$D_{10}^2 = \epsilon_{\mu,0}^2 + \epsilon_{\mu,1}^2 \quad , \tag{24}$$

and

$$D_{20}^2 = \epsilon_{\mu,0}^2 + \epsilon_{\mu,2}^2 \quad , \tag{25}$$

where $\epsilon_{\mu,0}$ is the external mean error of μ_0, defined by analogy to $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ (Eq.(5)):

$$\epsilon_{\mu,0}^2 = < (\mu_0 - \mu_w)^2 > = < (\Delta\mu_{0w})^2 > \quad . \tag{26}$$

The Eqs.(23), (24), and (25) provide three equations for the three unknowns $\epsilon_{\mu,1}, \epsilon_{\mu,2}$, and$\epsilon_{\mu,0}$. The solutions are:

$$\begin{aligned} \epsilon_{\mu,1}^2 &= \frac{1}{2}\,(D_{21}^2 + D_{10}^2 - D_{20}^2) \quad , \\ \epsilon_{\mu,2}^2 &= \frac{1}{2}\,(D_{21}^2 + D_{20}^2 - D_{10}^2) \quad , \\ \epsilon_{\mu,0}^2 &= \frac{1}{2}\,(D_{20}^2 + D_{10}^2 - D_{21}^2) \quad . \end{aligned} \tag{27}$$

The error $\epsilon_{\mu,0}$ of μ_0 provides also the combined positional errors of the catalogues by

$$\epsilon_{x,1}^2(T_1) + \epsilon_{x,2}^2(T_2) = \epsilon_{\mu,0}^2(T_2 - T_1)^2 \, . \tag{28}$$

6. The K method

In the K method, we make use of the correlation between $\Delta x_{21}(t)$ and $\Delta\mu_{21}$. We compute for each star the product

$$K_{21}(t) = \Delta x_{21}(t)\Delta\mu_{21} \quad , \tag{29}$$

where $\Delta x_{21}(t)$ is given by Eq.(9) and $\Delta\mu_{21}$ by Eq.(1). We then specify $t = T_1$, for example, divide by $T_2 - T_1$, and take the mean over n stars common to both catalogues:

$$\begin{aligned} &< K_{21}(T_1)/(T_2 - T_1) > \\ &= < \Delta x_{21}(T_1)\Delta\mu_{21}/(T_2 - T_1) > \\ &= < (x_2(T_1) - x_1(T_1))(\mu_2 - \mu_1)/(T_2 - T_1) > \\ &= < (x_2(T_2) - \mu_2(T_2 - T_1) - x_1(T_1))(\mu_2 - \mu_1) \\ &\quad /(T_2 - T_1) > \\ &= < ((x_2(T_2) - x_w(T_2)) - (x_1(T_1) - x_w(T_1)) \\ &\quad -(\mu_2 - \mu_w)(T_2 - T_1)) \\ &\quad ((\mu_2 - \mu_w) - (\mu_1 - \mu_w))/(T_2 - T_1) > \\ &= < (\Delta x_{2w}(T_2) - \Delta x_{1w}(T_1) - \Delta\mu_{2w}(T_2 - T_1)) \\ &\quad (\Delta\mu_{2w} - \Delta\mu_{1w})/(T_2 - T_1) > \\ &= - < (\Delta\mu_{2w})^2 > = -\epsilon_{\mu,2}^2 \, . \end{aligned} \tag{30}$$

All the averages of the mixed terms, $< \Delta x_{2w}(T_2)\Delta\mu_{2w} >$, $< \Delta x_{1w}(T_1)\Delta\mu_{2w} >, < \Delta x_{2w}(T_2)\Delta\mu_{1w} >$, $< \Delta x_{1w}(T_1)\Delta\mu_{1w} >$, and $< \Delta\mu_{2w}\Delta\mu_{1w} >$, vanish because of our statistical assumptions about the independence of the Catalogues 1 and 2 and the non-correlation of $\Delta x_{kw}(T_k)$ and $\Delta\mu_{kw} (k = 1,2)$ at the central epoch T_k. These assumptions should still be valid if we include into the averages an individual denominator $T_2 - T_1$. In a similar way, we obtain for $t = T_2$:

$$\begin{aligned} < K_{21}(T_2)/(T_2 - T_1) > &= < \Delta x_{21}(T_2)\Delta\mu_{21}/(T_2 - T_1) > \\ &= < (\Delta\mu_{1w})^2 > \\ &= \epsilon_{\mu,1}^2 \quad . \end{aligned} \tag{31}$$

Hence again, we have determined $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ separately. Instead of Eq.(30), we may use Eqs.(23) and (31) for deriving $\epsilon_{\mu,1}$:

$$\epsilon^2_{\mu,2} = D^2_{21} - \epsilon^2_{\mu,1} \quad . \tag{32}$$

7. Comparison of the three methods

As already mentioned, the three methods are strictly identical in principle. We shall prove this, as an example, for the results of the μ_0 method and the K method for $\epsilon_{\mu,1}$. According to Eq.(27), the contribution of a star to $\epsilon^2_{\mu,1}$ within the μ_0 method is given by

$$\begin{aligned}
&\frac{1}{2}\left((\mu_2-\mu_1)^2 + (\mu_1-\mu_0)^2 - (\mu_2-\mu_0)^2\right) \\
&= \mu_1^2 - \mu_1\mu_2 - \mu_1\mu_0 + \mu_2\mu_0 \\
&= (\mu_2-\mu_1)(\mu_0-\mu_1) = \Delta\mu_{21}\left(\frac{x_2(T_2)-x_1(T_1)}{T_2-T_1} - \mu_1\right) \\
&= \Delta\mu_{21}\left(\frac{x_2(T_2)-x_1(T_2)+\mu_1(T_2-T_1)}{T_2-T_1} - \mu_1\right) \\
&= \Delta\mu_{21}\Delta x_{21}(T_2)/(T_2-T_1) \quad .
\end{aligned} \tag{33}$$

The last line of Eq.(33) represents just the contribution of the star to $\epsilon^2_{\mu,1}$ within the K method, according to Eq.(31). Hence both methods give exactly the same results. The corresponding proof of the equivalence of the t_{min} method and the K method (and hence of the t_{min} method and the μ_0 method) is longer and therefore not given here.

Because of the equivalence of the three methods, the aspect under consideration should decide about the choice of the method. The t_{min} method gives, for our taste, the most vivid impression why $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ can be determined separately at all. The μ_0 method is mathematically the simplest one, and allows easily a generalisation of the method to many catalogues (see Section 11). The K method is, according to our experience, most easily used in practical applications.

8. The statistical uncertainty of the results

Due to the finite number n of stars which can be included into the comparison of the two catalogues, the results for $\epsilon_{\mu,1}$ etc. are statistically uncertain. We make the 'Gedanken-Experiment' that we envisage M equivalent pairs (with the index $m = 1,\ldots,M$) of catalogues. The pair m gives the result $(\epsilon^2_{\mu,1})_m$ etc. . We measure now the statistical uncertainty of $\epsilon^2_{\mu,1}$ by

$$S(\epsilon^2_{\mu,1}) = \left(\frac{1}{M}\sum_{m=1}^{M}((\epsilon^2_{\mu,1})_m - (\epsilon^2_{\mu,1})_w)^2\right)^{1/2} \quad , \tag{34}$$

where $(\epsilon^2_{\mu,1})_w$ is the true value of $\epsilon^2_{\mu,1}$ in the ensemble of the pairs, which may be approximately determined by the mean over the M values of $(\epsilon^2_{\mu,1})_m$. Here and in the following, we assume $M \gg 1$ and $n \gg 1$, so that e.g. $n-1$ can be approximated by n. S represents the dispersion of $(\epsilon^2_{\mu,1})_m$, not of $(\epsilon_{\mu,1})_m$ itself. It turns out that it is mathematically easier to determine S than a similarly defined disperson s of $(\epsilon_{\mu,1})_m$ itself:

$$s(\epsilon_{\mu,1}) = \left(\frac{1}{M}\sum_{m=1}^{M}((\epsilon_{\mu,1})_m - (\epsilon_{\mu,1})_w)^2\right)^{1/2} \quad . \tag{35}$$

For small values of s and $\sqrt{S}$ with respect to $(\epsilon_{\mu,1})_w$, we have

$$S(\epsilon^2_{\mu,1}) = 2\,(\epsilon_{\mu,1})_w\, s(\epsilon_{\mu,1}) \quad . \tag{36}$$

In order to derive $S(\epsilon^2_{\mu,1})$, we use Eq.(27) of the μ_0 method and make partially use of Eq.(33):

$$\begin{aligned}
(\epsilon^2_{\mu,1})_m &= <\frac{1}{2}((\mu_2-\mu_1)^2 + (\mu_1-\mu_0)^2 - (\mu_2-\mu_0)^2)>_m \\
&= <(\mu_2-\mu_1)(\mu_0-\mu_1)>_m \\
&= <((\mu_2-\mu_w)-(\mu_1-\mu_w)) \\
&\quad ((\mu_0-\mu_w)-(\mu_1-\mu_w))>_m \\
&= <(\Delta\mu_{2w}-\Delta\mu_{1w})(\Delta\mu_{0w}-\Delta\mu_{1w})>_m \\
&= <(\Delta\mu_{1w})^2>_m + <\Delta\mu_{2w}\Delta\mu_{0w}>_m \\
&\quad - <\Delta\mu_{1w}\Delta\mu_{0w}>_m - <\Delta\mu_{2w}\Delta\mu_{1w}>_m \; .
\end{aligned} \tag{37}$$

We have used the index m to indicate that the result of Eq.(37) is valid for an individual pair m. While the mixed terms would vanish on average over many pairs and the quadratic term would approach $(\epsilon^2_{\mu,1})_w$, all the four terms in Eq.(37) produce a statistical noise in $(\epsilon^2_{\mu,1})_m$.

For deriving S, we now assume that all the differences $\Delta\mu_{1w}, \Delta\mu_{2w}$, and $\Delta\mu_{0w}$ have a gaussian distribution with mean zero and dispersion $(\epsilon_{\mu,1})_w$, $(\epsilon_{\mu,2})_w$, and $(\epsilon_{\mu,0})_w$. The term $<(\Delta\mu_{1w})^2>$ has then (for large n and M) a mean value of $(\epsilon^2_{\mu,1})_w$ and a dispersion $disp$ around that mean of

$$disp\,(<(\Delta\mu_{1w})^2>_m) = \left(\frac{2}{n}\right)^{1/2}(\epsilon^2_{\mu,1})_w \quad , \tag{38}$$

as it is well-known from the χ^2 distribution. As an example of one of the three mixed terms in Eq.(37), we treat $C_{21} = <\Delta\mu_{2w}\Delta\mu_{1w}>_m$. The term C_{21} has the mean zero. The distribution function of $\Delta\mu_{1w}\Delta\mu_{2w}$ is given by

$$\begin{aligned}
f(\Delta\mu_{1w}\Delta\mu_{2w}) &= K_0\,(\Delta\mu_{1w}\Delta\mu_{2w}/((\epsilon_{\mu,1})_w(\epsilon_{\mu,2})_w)) \\
&\quad /\,(\pi\,(\epsilon_{\mu,1})_w(\epsilon_{\mu,2})_w) \quad ,
\end{aligned} \tag{39}$$

where K_0 is one of the modified Bessel functions. The dispersion of C_{21} can be shown to be

$$disp\,(<\Delta\mu_{2w}\Delta\mu_{1w}>_m) = \frac{1}{n^{1/2}}\,(\epsilon_{\mu,1})_w(\epsilon_{\mu,2})_w \quad . \tag{40}$$

Similar equations can be derived for the other two mixed terms:

$$disp\,(<\Delta\mu_{1w}\Delta\mu_{0w}>_m) = \frac{1}{n^{1/2}}\,(\epsilon_{\mu,0})_w(\epsilon_{\mu,1})_w \quad , \tag{41}$$

$$disp\,(<\Delta\mu_{2w}\Delta\mu_{0w}>_m) = \frac{1}{n^{1/2}}\,(\epsilon_{\mu,0})_w(\epsilon_{\mu,2})_w \quad . \qquad (42)$$

The square of the total dispersion of the sum of the four terms of Eq.(37) is obtained, as usual, by adding the squares of the dispersions of each of the four terms, since the mixed products among the four terms vanish on average. We now suppress the index w of ϵ, thereby replacing the true values of ϵ by their empirically determined values:

$$\begin{aligned} S^2(\epsilon^2_{\mu,1}) &= (disp\,((\epsilon^2_{\mu,1})_m))^2 \\ &= \frac{1}{n}(2\epsilon^4_{\mu,1} + \epsilon^2_{\mu,1}\epsilon^2_{\mu,2} + \epsilon^2_{\mu,0}\epsilon^2_{\mu,1} + \epsilon^2_{\mu,0}\epsilon^2_{\mu,2})\,, \\ S^2(\epsilon^2_{\mu,2}) &= \frac{1}{n}(2\epsilon^4_{\mu,2} + \epsilon^2_{\mu,1}\epsilon^2_{\mu,2} + \epsilon^2_{\mu,0}\epsilon^2_{\mu,1} + \epsilon^2_{\mu,0}\epsilon^2_{\mu,2})\,, \\ S^2(\epsilon^2_{\mu,0}) &= \frac{1}{n}(2\epsilon^4_{\mu,0} + \epsilon^2_{\mu,1}\epsilon^2_{\mu,2} + \epsilon^2_{\mu,0}\epsilon^2_{\mu,1} + \epsilon^2_{\mu,0}\epsilon^2_{\mu,2})\,. \qquad (43) \end{aligned}$$

From Eqs.(43), it is obvious that a mean error, say $\epsilon_{\mu,1}$, can be only reliably determined if none of the other two errors (here $\epsilon_{\mu,2}$ and $\epsilon_{\mu,0}$) is much larger than this value (here $\epsilon_{\mu,1}$). In many applications, the mean errors of the three proper motions will be roughly comparable. To illustrate a typical case, we assume $\epsilon_{\mu,1} = \epsilon_{\mu,2} = \epsilon_{\mu,0}$. The statistical uncertainty is then the same for all the derived mean errors. For $n = 100$ stars, we have $S(\epsilon^2_\mu)/\epsilon^2_\mu = 22\%$ and $s(\epsilon_\mu)\ /\ \epsilon_\mu = 11\%$; for $n = 1000$ stars, the corresponding uncertainties are already as small as 7.1% and 3.5%. It should be remembered, however, that our estimates of the statistical uncertainty of ϵ_μ are based on the assumption of a gaussian distribution of the proper motion differences $\Delta\mu_{1w}$, $\Delta\mu_{2w}$, $\Delta\mu_{0w}$. In reality, the uncertainty may be somewhat larger.

9. Errors varying from star to star

Up to now, we have assumed that neither the internal mean errors of μ and x in a catalogue nor the external errors vary from star to star. In most applications, this would mean that we determine an rms average over the individual mean errors of the n stars common to both catalogues.

Our method cannot be easily generalized to the case of mean errors which vary from star to star. At least, we would need information on the probable relations between internal errors σ_i and external errors ϵ_i, like $\epsilon_i = const.\sigma_i$ or $\epsilon^2_i = \sigma^2_i + const.$. In practice, the most reasonable procedure seems to be to apply our method to rather homogeneous subgroups of stars which are selected in such a way that their internal errors σ_i are nearly the same within each catalogue (but are allowed to differ between the two catalogues). Especially, we should remove in any case stars with exceptionally large internal errors from our sample of n common stars, before we apply our method.

10. Numerical results for simulated catalogues

We have tested the basic validity and some properties of our methods on numerically simulated catalogues. In these artificial catalogues, the distribution of the differences $\Delta\mu_{1w}$, $\Delta\mu_{2w}$, $\Delta\mu_{0w}$, $\Delta x_{1w}(T_1)$, $\Delta x_{2w}(T_2)$ are gaussian functions with a mean zero and a chosen dispersion $\sigma_{\mu,1}$, $\sigma_{\mu,2}$, $\sigma_{\mu,0}$, $\sigma_{x,1}$, $\sigma_{x,2}$. From these distributions, we draw simulated catalogues, each containing n stars, by using a random number generator. We then apply our methods, and test how much the inserted internal dispersions (σ) agree with the output of our methods, the determined external errors (ϵ). The simulation allows us also to study experimentally the statistical uncertainty of our results, e.g. as a function of n, the number of stars used. The number of pairs of catalogues which we use to derive averages and dispersions for our results, is $M = 10^4$ for Tables 1–3.

The results of our numerical simulations agree perfectly with the predictions of our method. The experiments confirm that the three methods (t_{min}, μ_0, K) give exactly the same answers. Hence it is not necessary to list the results of the three methods separately in the tables. In Tab.1–3, we give, beside the input data, the quadratic averages over the M pairs, such as $<(\epsilon^2_{\mu,1})_m>$, as the typical empirical result for ϵ^2_μ. They always agree with the input values of σ^2_μ, thereby confirming that our methods are unbiased with respect to ϵ^2_μ, even for small values of n. We list further the empirically determined dispersions S of ϵ^2_μ within the sample of M pairs. The agreement between the experimentally determined values of S and the analytically predicted dispersion (Eqs.(43) is also very good. In addition to $<\epsilon^2_\mu>$ and S, we give in Tab.1 and 2 $<(\epsilon_\mu)_m>$ and the corresponding dispersion s of ϵ_μ (not from Eq.(36), but directly determined from the scatter of $(\epsilon_\mu)_m$ around $<(\epsilon_\mu)_m>$). $<(\epsilon_\mu)_m>$ is biased to smaller values than σ_μ, but only significantly for very small n; it is not equal to $(<(\epsilon_{\mu,m})^2>)^{1/2}$, because the distribution function of $\epsilon_{\mu,m}$ is skew for small n. It is not possible to give meaningful results for $<(\epsilon_\mu)_m>$ or s, if ϵ_μ is very small with respect to the other mean errors or even zero (such as $\epsilon_{\mu,0}$ in Tab.3). In such cases, the statistical noise of our method produces (correctly) in some ($\sim M/2$) cases a negative value for ϵ^2_μ. For $<\epsilon^2_\mu>$ and S, we can use these negative values of ϵ^2_μ without any problem, while that is obviously not possible for $<\epsilon_\mu>$ or s. We use, rather arbitrarily, $(\epsilon_\mu)_m = -|(\epsilon^2_\mu)_m|^{1/2}$ for negative values of $(\epsilon^2_\mu)_m$. The units used in our Tables 1,2, and 3 are mas/year or (mas/year)^2, respectively, but this does not really matter when comparing the input data with the output results.

The input data for the two simulated catalogues, used in Table 1, are somewhat similar to the internal mean errors of the basic FK5 (Catalogue 1) and to what we might roughly expect for the bright stars in the final Hipparcos catalogue (Catalogue 2). The dispersion of μ_0, namely $\sigma_{\mu,0} = 0.5$ mas/year, is based on $\sigma_{x,1}(T_1) = 20$ mas, neg-

Table 1. Results for numerically simulated catalogues (1st example)

Input:	$\sigma^2_{\mu,1}$		$\sigma^2_{\mu,2}$		$\sigma^2_{\mu,0}$		$\sigma_{\mu,1}$		$\sigma_{\mu,2}$		$\sigma_{\mu,0}$	
	0.64		1.00		0.25		0.80		1.00		0.50	
Output:	$< \epsilon^2_{\mu,1} >$	$S(\epsilon^2_{\mu,1})$	$< \epsilon^2_{\mu,2} >$	$S(\epsilon^2_{\mu,2})$	$< \epsilon^2_{\mu,0} >$	$S(\epsilon^2_{\mu,0})$	$< \epsilon_{\mu,1} >$	$s(\epsilon_{\mu,1})$	$< \epsilon_{\mu,2} >$	$s(\epsilon_{\mu,2})$	$< \epsilon_{\mu,0} >$	$s(\epsilon_{\mu,0})$
$n[stars]$												
1	0.63	1.35	1.00	1.73	0.26	1.09	(0.43	0.83	0.63	0.86	0.19	0.81)
10	0.64	0.43	1.00	0.55	0.25	0.35	(0.73	0.33	0.95	0.29	0.35	0.46)
20	0.64	0.31	1.00	0.39	0.25	0.24	(0.77	0.21	0.98	0.20	0.41	0.35)
50	0.64	0.19	1.00	0.25	0.25	0.15	0.79	0.12	0.99	0.13	(0.46	0.21)
100	0.64	0.14	1.00	0.17	0.25	0.11	0.80	0.09	1.00	0.09	(0.49	0.13)
200	0.64	0.10	1.00	0.12	0.25	0.08	0.80	0.06	1.00	0.06	0.49	0.08
500	0.64	0.06	1.00	0.08	0.25	0.05	0.80	0.04	1.00	0.04	0.50	0.05
1000	0.64	0.04	1.00	0.05	0.25	0.03	0.80	0.03	1.00	0.03	0.50	0.03

Table 2. Results for numerically simulated catalogues (2nd example)

Input:	$\sigma^2_{\mu,1}$		$\sigma^2_{\mu,2}$		$\sigma^2_{\mu,0}$		$\sigma_{\mu,1}$		$\sigma_{\mu,2}$		$\sigma_{\mu,0}$	
	1.00		1.00		1.00		1.00		1.00		1.00	
Output:	$< \epsilon^2_{\mu,1} >$	$S(\epsilon^2_{\mu,1})$	$< \epsilon^2_{\mu,2} >$	$S(\epsilon^2_{\mu,2})$	$< \epsilon^2_{\mu,0} >$	$S(\epsilon^2_{\mu,0})$	$< \epsilon_{\mu,1} >$	$s(\epsilon_{\mu,1})$	$< \epsilon_{\mu,2} >$	$s(\epsilon_{\mu,2})$	$< \epsilon_{\mu,0} >$	$s(\epsilon_{\mu,0})$
$n[stars]$												
1	0.98	2.21	1.00	2.22	1.01	2.26	(0.52	1.07	0.53	1.07	0.53	1.08)
10	1.00	0.70	0.99	0.71	1.00	0.72	(0.91	0.44	0.90	0.45	0.91	0.45)
100	1.00	0.23	1.00	0.22	1.00	0.22	0.99	0.11	0.99	0.11	0.99	0.11
1000	1.00	0.07	1.00	0.07	1.00	0.07	1.00	0.04	1.00	0.03	1.00	0.04

Table 3. Results for numerically simulated catalogues (3rd example)

Input:	$\sigma^2_{\mu,1}$		$\sigma^2_{\mu,2}$		$\sigma^2_{\mu,0}$		$\sigma_{\mu,1}$		$\sigma_{\mu,2}$		$\sigma_{\mu,0}$	
	1.00		1.00		0.00		1.00		1.00		0.00	
Output:	$< \epsilon^2_{\mu,1} >$	$S(\epsilon^2_{\mu,1})$	$< \epsilon^2_{\mu,2} >$	$S(\epsilon^2_{\mu,2})$	$< \epsilon^2_{\mu,0} >$	$S(\epsilon^2_{\mu,0})$	$< \epsilon_{\mu,1} >$	$s(\epsilon_{\mu,1})$	$< \epsilon_{\mu,2} >$	$s(\epsilon_{\mu,2})$	$< \epsilon_{\mu,0} >$	$s(\epsilon_{\mu,0})$
$n[stars]$												
1	0.99	1.71	0.99	1.72	0.01	1.00	(0.64	0.85	0.63	0.85	0.00	0.80)
10	1.00	0.55	1.00	0.55	0.00	0.32	(0.96	0.29	0.96	0.29	0.01	0.50)
100	1.00	0.17	1.00	0.17	0.00	0.10	1.00	0.09	1.00	0.09	(0.00	0.28)
1000	1.00	0.05	1.00	0.05	0.00	0.03	1.00	0.03	1.00	0.03	(0.00	0.16)

ligible $\sigma_{x,2}(T_2)$, and $T_2 - T_1 = 40$ years. We emphasize, however, that both catalogues used in Table 1 are fully artificial, and should not be identified in any sense with the real catalogues.

In the Tables 2 and 3, we give the results of numerical simulations, in which we put in $\sigma_{\mu,1} = \sigma_{\mu,2} = 1$ mas/year and $\sigma_{\mu,0} = 1$ or 0 mas/year. The case $\sigma_{\mu,2} = \sigma_{\mu,0} = 0$ would reproduce only results known already from the χ^2 distribution.

Especially the results presented in Tables 1 and 2 confirm our conclusion of Section 8: If $\epsilon_{\mu,1}, \epsilon_{\mu,2}, \epsilon_{\mu,0}$ are roughly equal, they can be all empirically determined with a relative statistical uncertainty $s(\epsilon_\mu)/\epsilon_\mu$ of a few percent, if about $n = 1000$ stars are available.

11. Generalisation for many catalogues

The method presented above is based on the availability of two independent proper motions, μ_1 and μ_2, and two independent positions, $x_1(T_1)$ and $x_2(T_2)$. A closer inspection of the method reveals that as long as Eq.(3) is valid, the positions $x_k(T_k)$ and the proper motions $\mu_k (k = 1, 2)$

have not to be provided by a common catalogue k, but may have been determined independently from each other.

We may now generalize the situation: Let us assume that we have available a number of independent catalogues which together provide for each star N_μ proper motions $\mu_j (j = 1, \dots, N_\mu)$ and N_x positions $x_\ell(T_\ell)(\ell = 1, \dots, N_x)$. In general, the epochs T_ℓ should differ from each other. The method discussed above corresponds to $N_\mu = 2$ and $N_x = 2$. This means that a catalogue like the FK5 contributes both to N_μ and N_x, while a purely positional catalogue contributes only to N_x.

We are now investigating whether or not we can determine the external mean errors $\epsilon_{\mu,j}$ and $\epsilon_{x,\ell}$ of the $N_\mu + N_x$ proper motions and positions separately (For writing economy, we use here $\epsilon_{x,\ell} \equiv \epsilon_{x,\ell}(T_\ell)$.). In order to do so we generalize the μ_0 method, presented in Section 5. From two positions, $x_\ell(T_\ell)$ and $x_{\ell'}(T_{\ell'})$, we derive a proper motion

$$\mu_{(\ell'\ell)} = (x_{\ell'}(T_{\ell'}) - x_\ell(T_\ell))/(T_{\ell'} - T_\ell) \quad , \qquad (44)$$

and we form the rms value of the difference of two proper motions $\mu_{(\ell'\ell)}$ and $\mu_{(\ell'''\ell'')}$:

$$D^2_{(\ell'''\ell'')(\ell'\ell)} = < (\mu_{(\ell'''\ell'')} - \mu_{(\ell'\ell)})^2 > \quad . \qquad (45)$$

If the positions are fully distinct, e.g. $(\ell'''\ell'')(\ell'\ell) = (43)(21)$, then the expectation value of $D^2_{(\ell'''\ell'')(\ell'\ell)}$ is given by

$$D^2_{(\ell'''\ell'')(\ell'\ell)} = \frac{\epsilon^2_{x,\ell} + \epsilon^2_{x,\ell'}}{(T_{\ell'} - T_\ell)^2} + \frac{\epsilon^2_{x,\ell''} + \epsilon^2_{x,\ell'''}}{(T_{\ell'''} - T_{\ell''})^2} \quad . \qquad (46)$$

If the two proper motions have one position in common, then the two derived proper motions are correlated (but still independent), and we obtain

$$D^2_{(\ell''\ell')(\ell'\ell)} = \frac{\epsilon^2_{x,\ell}}{(T_{\ell'} - T_\ell)^2} + \epsilon^2_{x,\ell'} \left(\frac{1}{T_{\ell'} - T_\ell} + \frac{1}{T_{\ell''} - T_{\ell'}} \right)^2 + \frac{\epsilon^2_{x,\ell''}}{(T_{\ell''} - T_{\ell'})^2} \quad . \qquad (47)$$

We may also form the rms difference between a derived proper motion $\mu_{(\ell'\ell)}$ and a directly given one, μ_j,

$$D^2_{j(\ell'\ell)} = < (\mu_j - \mu_{(\ell'\ell)})^2 > = \epsilon^2_{\mu,j} + \frac{\epsilon^2_{x,\ell} + \epsilon^2_{x,\ell'}}{(T_{\ell'} - T_\ell)^2} \quad , \qquad (48)$$

or between two directly given proper motions μ_j and $\mu_{j'}$:

$$D^2_{j'j} = < (\mu_{j'} - \mu_j)^2 > = \epsilon^2_{\mu,j} + \epsilon^2_{\mu,j'} \quad . \qquad (49)$$

The set of Eqs.(46), (47), (48), and (49), for as many pairs of proper motions as possible, have now to be solved for the $N_\mu + N_x$ unknowns $\epsilon_{\mu,j} (j = 1, \dots, N_\mu)$ and $\epsilon_{x,\ell}(\ell = 1, \dots, N_x)$. The question is then whether or not we have enough independent equations for doing so, i.e. at least $N_\mu + N_x$ equations. Equations of the type of Eq.(48) can be used, in spite of the fact that $\mu_{(\ell'\ell)}$ and $\mu_{(\ell''\ell')}$ are correlated. But we should not use 'non-independent' proper motions, which can be strictly derived from proper motions which we have already used as independent proper motions. An example of such a 'non-independent' proper motion is $\mu_{(\ell''\ell)}$, if we have already used $\mu_{(\ell'\ell)}$ and $\mu_{(\ell''\ell')}$, since

$$\mu_{(\ell''\ell)}(T_{\ell''} - T_\ell) = \mu_{(\ell'\ell)}(T_{\ell'} - T_\ell) + \mu_{(\ell''\ell')}(T_{\ell''} - T_{\ell'}) \,. \qquad (50)$$

Let us start with the easiest case of N_μ proper motions alone, i.e. $N_x = 0$. The smallest solvable value of N_μ is then 3: we form 3 pairs of Eq.(49) for $j'j = 21$, 31, and 32, and solve these 3 equations for the 3 unknowns $\epsilon_{\mu,1}, \epsilon_{\mu,2}, \epsilon_{\mu,3}$. If we have more than 3 proper motions, i.e. $N_\mu = 4$ or higher, we have more independent equations (namely $N_\mu(N_\mu - 1)/2$, i.e. the number of distinct pairs which can be formed out of N_μ proper motions) than unknowns (N_μ). In this case, we have to solve this overdetermined set of equations by a 'best-fit' procedure, e.g. a kind of least-squares method.

The other extreme situation is $N_\mu = 0$, i.e. we have N_x positions only. From N_x positions with distinct epochs $T_\ell(\ell = 1, \dots, N_x)$, we can derive $N_x(N_x - 1)/2$ proper motions, but only $N_x - 1$ of them are independent from each other. From these $N_x - 1$ independent proper motions, we can form $X = (N_x - 1)(N_x - 2)/2$ pairs of proper motions, and hence X equations of the type of Eq.(46) or (47). Since we have N_x unknowns, $\epsilon_{x,\ell}(\ell = 1, \dots, N_x)$, we need $X \geq N_x$ equations. This relation is fulfilled for $N_x \geq 5$. For $N_x = 4$, we have only 3 pairs and hence 3 equations. For $N_x = 5$, we have already 4 pairs and 6 equations. Therefore, it is possible to determine all $\epsilon_{x,\ell}$ separately for $N_x \geq 5$, even if no proper motions are directly given ($N_\mu = 0$).

$N_x = 1$ and $N_x = 2$ are special cases. $N_x = 1$ is essentially equivalent to $N_x = 0$, since we can neither determine ϵ_x nor are the positions helpful for determining ϵ_μ. In the case of $N_x = 2$, we can derive one proper motion from Eq.(44). Together with N_μ direct proper motions, we have then $N_\mu + 1$ pairs of independent proper motions. For $N_\mu \geq 2$, we can solve for all $\epsilon_{\mu,j}(j = 1, \dots, N_\mu)$ and for $\epsilon^2_{x,1} + \epsilon^2_{x,2}$; but it is impossible to separate $\epsilon_{x,1}$ from $\epsilon_{x,2}$, no matter how large N_μ is.

The most general case it $N_\mu \geq 0$ and $N_x \geq 3$. Then we have $Y = N_\mu + N_x - 1$ independent proper motions. From these Y proper motions, we can form $Z = Y(Y-1)/2 = (N_\mu + N_x - 1)(N_\mu + N_x - 2)/2$ pairs, and hence Z equations of the type of Eqs.(45–49). In order to solve for all the $N_\mu + N_x$ unknowns separately, it is necessary to have at least as many equations, $Z \geq N_\mu + N_x$. This condition is fulfilled for $N_\mu + N_x \geq 5$. In summary, we can solve for $\epsilon_{\mu,j}(j = 1, \dots, N_\mu \geq 0)$ and $\epsilon_{x,\ell}(\ell = 1, \dots, N_x \geq 3)$, if $N_x = 3$ and $N_\mu \geq 2$, or $N_x = 4$ and $N_\mu \geq 1$, or $N_x \geq 5$ and $N_\mu \geq 0$.

In practice, it will be necessary to modify the procedure slightly in some cases. For example, if the epochs

T_ℓ and $T_{\ell'}$ of two positions are too close together, it is not advisable to use the proper motion $\mu_{(\ell'\ell)}$. If we have enough other data available, we may first solve the problem without using $x_\ell(T_\ell)$ or $x_{\ell'}(T_{\ell'})$. Then we select the best proper motion, say μ_j with its now known external error $\epsilon_{\mu,j}$, and form differences of positions at a common epoch, say at T_ℓ:

$$\begin{aligned} d^2_{\ell'\ell}(T_\ell) &= <(x_{\ell'}(T_{\ell'}) + \mu_j(T_\ell - T_{\ell'}) - x_\ell(T_\ell))^2> \\ &= \epsilon^2_{x,\ell} + \epsilon^2_{x,\ell'} + \epsilon^2_{\mu,j}(T_\ell - T_{\ell'})^2 \quad , \end{aligned} \tag{51}$$

and similarly $d^2_{\ell''\ell}(T_\ell)$ and $d^2_{\ell''\ell'}(T_\ell)$. Then these equations can be solved for $\epsilon_{x,\ell}$, $\epsilon_{x,\ell'}$, and $\epsilon_{x,\ell''}$ separately. Fully degenerated cases, like $T_\ell = T_{\ell'} = T_{\ell''}$, are of academic interest only, even if they are more easily solvable.

Finally, we should remark that it is always necessary (and possible) to reduce all the data, i.e. the μ_j and the $x_\ell(T_\ell)$, to a common system (see Section 3), before we determine the external mean errors $\epsilon_{\mu,j}$ and $\epsilon_{x,\ell}$ of the data by the methods described above.

12. Astrometric binaries and conclusions

One of the main purposes of the method presented above is to allow us to compare the derived external errors $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$ of the two proper motions (μ_1 and μ_2) with their predicted internal errors, $\sigma_{\mu,1}$ and $\sigma_{\mu,2}$. If the internal and external errors do agree, we are fully satisfied. If ϵ_μ is significantly larger than σ_μ, we are inclined to blame the catalogues which provide these internal errors σ_μ for having overestimated the internal accuracy of the proper motions. This judgement will be correct in some cases. In other cases, however, we have to blame nature for contributing a 'cosmic dispersion' or, more honestly, to realize that one of our basic assumptions is not valid.

The most basic assumption of our method (see Section 3) is that the stars under consideration move linearly in time on straight lines (Eq.(3)). Single stars will do so to a very high degree of approximation in most applications. Most stars, however, are members of binaries, and do not follow Eq.(3). We should therefore select single stars for an application of our method. Unfortunately, a high number of apparently 'single' stars which we have selected may be in reality members of binaries, especially of hithertoo undetected 'astrometric binaries'. The measured brighter member of such a double star, or (more precisely) the photocenter of the binary system, will show a (non-linear) orbital motion around the center of mass of the system, which still moves linearily according to Eq.(3). Some catalogues, like Hipparcos, will measure mainly 'instantaneous' positions and proper motions which include the orbital displacement and the orbital motion around the center of mass at the time of measurement. Other catalogues, like the FK5, average over long periods of time (more than a century in the case of FK5); their data will in many cases represent the 'long-term' center-of-mass motion of the binary system. If we now form the difference $\Delta\mu_{21} = \mu_2 - \mu_1$ between either two instantaneous proper motions or between one instantaneous proper motion and one long-term one, the difference $\Delta\mu_{21}$ will contain, in addition to the real measuring errors $\Delta\mu_{2w}$ and $\Delta\mu_{1w}$, a contribution of the orbital motion of the *a*strometric *b*inary, which we call $\Delta\mu_{ab}$. Our method, in the form presented above, will then (correctly) include $\Delta\mu_{ab}$ into the deviations from linear motion, represented formally by $\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$. The formal results of our method ($\epsilon_{\mu,1}$ and $\epsilon_{\mu,2}$) will, in contrast to our basic aim, not represent the true external errors of μ_1 and μ_2, which we shall denote by $\tilde{\epsilon}_{\mu,1}$ and $\tilde{\epsilon}_{\mu,2}$. In general, we will find $\epsilon_\mu > \tilde{\epsilon}_\mu$, since

$$D^2_{21} = <(\Delta\mu_{21})^2> = \tilde{\epsilon}^2_{\mu,1} + \tilde{\epsilon}^2_{\mu,2} + \tilde{\epsilon}^2_{ab} = \epsilon^2_{\mu,1} + \epsilon^2_{\mu,2} \; , \tag{52}$$

where $\tilde{\epsilon}_{ab}$ is the rms value of $\Delta\mu_{ab}$:

$$\tilde{\epsilon}^2_{ab} = <(\Delta\mu_{ab})^2> \quad . \tag{53}$$

How much of $\tilde{\epsilon}_{ab}$ is taken over by $\epsilon_{\mu,1}$ or $\epsilon_{\mu,2}$, depends on the details of the situation, e.g. on the periods of the astrometric binaries.

For illustration, we shall give an artificial (but rather fair) example: We take a basic FK5 star of spectral type A at a distance of $r = 100$ pc (corresponding to a parallax of $p = 10$ mas) and a mass of $m_1 = 2.5$ solar masses. As its companion we postulate an M star of $m_2 = 0.5$ solar masses. Furthermore, we assume an orbital period of $U = 50$ years, and a circular orbit for simplicity. The orbital velocity v_{1-2} of star 1 with respect to star 2 is then 11.7 km/s. The orbital velocity of star 1 (and approximately of the photo center) with respect to the center of mass is $v_{1-cms} = (m_2/(m_1+m_2))v_{1-2} = 1.9$ km/s. The rms value of the projection of the vector of v_{1-cms} on one (arbitrary) direction (thereby implicitely averaged over all directions and over time) is $\Delta v_{ab} = v_{1-cms}/\sqrt{3} = 1.1$ km/s.

It would be rather difficult to detect the binary nature of the system by direct measurements: magnitude difference between the two stars $\Delta m = 8^m$; maximum angular distance of the two stars $\rho = 0.''20$ for $p = 10$ mas; the typical deviation of an instantaneous radial velocity measurements of the bright star from its mean value is equal to $\Delta v_{ab} = 1.1$ km/s. The maximum angular distance of the bright star from the center of mass is 33 mas. Hence the FK5 would not find this variation in position. During 3 years, the deviation of the circular orbit from a fitting straight line would be less than 0.3 mas. Hence Hipparcos would also not detect the non-linearity in the motion of the star.

We now translate Δv_{ab} into a difference in proper motion:

$$\Delta\mu_{ab}[\mathrm{mas/a}] = \frac{1}{4.74}\Delta v_{ab}[\mathrm{km/s}]\, p\, [\mathrm{mas}] \quad . \tag{54}$$

For p = 10 mas, we get $\Delta\mu_{ab} = 2.4$ mas/year. Such a value is larger than the internal accuracy of the proper motions in the FK5 as well as in the expected Hipparcos catalogue.

Hence astrometric binaries may severely disturb the comparison of high-accuracy proper motions. While in many individual cases, the differences $\Delta\mu_{ab}$ cannot be safely determined (judged e.g. by a $3\sigma_\mu$ criterium), the effect of astrometric binaries should show up very significantly in the much better determined rms average over some hundred stars, as used in our method, e.g. in D_{21} (Eq.(22)).

Equation (54) offers, however, also a way to diminish the effect of astrometric binaries: $\Delta\mu_{ab}$ is, for a given value of Δv_{ab} in km/s, proportional to the parallax p. This means that the disturbing effect of undetected astrometric binaries will significantly go down with increasing distance. In our example, $\Delta\mu_{ab} = 0.24$ mas/year at $r = 1000$ pc ($p = 1$ mas). Hence one possible strategy to avoid as far as possible a perturbation of our method by astrometric binaries, is to select not only apparently single stars, but to use only the most distant 'single' stars for comparing the two catalogues. In general, however, the statistical effect of undetected astrometric binaries on proper motions will not be strictly declining proportional to the parallax, since both the fraction of undetected binaries among the 'single' stars and the value of Δv_{ab} in km/s shall increase with distance, thereby working in the opposite direction of the parallax.

In a subsequent paper, we shall discuss a modified version of our method which makes statistical allowance for the effect of undetected astrometric binaries on our estimates of the external accuracy of astrometric catalogues.

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