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Dynamical friction of massive objects in galactic centres

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ABSTRACT
Dynamical friction leads to an orbital decay of massive objects like young compact star clusters or Super Massive Black Holes in central regions of galaxies. The dynamical friction force can be well approximated by Chandrasekhar's standard formula, but recent investigations show, that corrections to the Coulomb logarithm are necessary. We propose to use the local scale length as maximum impact parameter and the orbital velocity of the massive object instead of the velocity dispersion of the background particles as typical encounter speed to determine the lower cutoff for a point like object. For extended objects the half mass radius is a good measure of the minimum impact parameter, which should be substituted by the smoothing length or half the cell size for numerical codes. Here we compare the new Coulomb logarithm with an independent numerical determination for different smoothing lengths and numerical codes. These numerical results are in good agreement with our theoretically demanded values. We also give a generalisation of the orbital evolution with a position dependent Coulomb logarithm in cuspy density profiles.

Key words: Stellar dynamics – Galaxies: kinematics and dynamics – Galaxy: center.

1 INTRODUCTION

In many different physical systems dynamical friction is the main process responsible for the orbital evolution of massive objects, like satellite galaxies in Dark Matter haloes, massive stars in star clusters, Super Massive Black Holes (SMBH) or star clusters is galaxy centres. Here we focus on the orbital evolution of SMBHs (settling to the centre after a galaxy merging process, see e.g. Milosavljević & Merritt (2001)) and of very young compact star clusters (which were born at larger distances, where the tidal field is weaker, e.g. Gerhard (2001)). The dynamical friction force of a massive object with mass $M_{\rm BH}$ and orbital velocity $v_{\rm BH}$ in a sea of lighter particles is usually determined by adopting locally a homogeneous back ground density ρ with isotropic Gaussian velocity distribution function with 1-dim. dispersion σ. The result is a drag force parallel to the motion, which is given by the standard formula of Chandrasekhar (Binney & Tremaine 1987)

$$\dot{v}_{\rm BH} = -4\pi G^2 \ln\Lambda M_{\rm BH}\frac{\chi}{v_{\rm BH}^2} \quad , \quad \text{where} \tag{1}$$

$$\chi = {\rm erf}(X) - \frac{2X}{\sqrt{\pi}}\exp(-X^2) \quad \text{and} \quad X = \frac{v_{\rm BH}}{\sqrt{2}\sigma} \tag{2}$$

The main uncertainties in the parameter dependence focus on the Coulomb logarithm $\ln\Lambda$, which gives the effective range of relevant impact parameters and is up to now a not well determined quantity. Since this formula is applied on wide ranges of parameters, it would be very useful to know the explicite parameter dependence of $\ln\Lambda$ instead of fitting a constant value for each application. The Coulomb logarithm depends on the maximum and minimum impact parameter $b_{\rm max}$ and $b_{\rm min}$, resp., and for point like objects on a_{90}, the average impact parameter for a 90° deflection in the 2-body encounters. This leads to a position dependence of $\ln\Lambda$, which affects the decay time $t_{\rm dc}$ (see Eq. A5) and which also reduce the circularisation of the orbits resolving a longstanding discrepancy between numerical results and theoretical expectations. Using the distance to the centre as maximum impact parameter is proposed by different authors (Tremaine 1976; Hashimoto et al. 2003).

In a recent investigation Spinnato et al. (2003) (hereafter SFP) determined in a large number of runs numerically the value of the Coulomb logarithm of a massive object moving through a background of stars with a power law cusp for different numerical codes and parameters. They found that $b_{\rm max}$ is systematically about a factor of two smaller than the initial distance R_0 to the centre. For a_{90} they also found a value significantly smaller than the gravitational influence radius $r_{\rm g} = GM_{\rm BH}/\sigma^2$. The smoothing length or the cell size ϵ of the codes works as a spatial extension of the massive object. SFP claim, that $b_{\rm min} = \epsilon$ for all types of code, a direct particle-particle code (hereafter PP), a tree code (Tree), and a particle-mesh code (PM). They also find a dependence of the Coulomb logarithm on the initial position, but for each numerical run they used a constant $\ln\Lambda$ to determine the best fit.

In a recent investigation Just & Peñarrubia (2002) discussed the effect of the inhomogeneity of the background distribution on dynamical friction. They find that the maximum impact parameters is given by the local scale length l determined by the density gradient, which is in the flat core region only given by the distance R to the centre, i.e.

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$$b_{\max} = l \equiv \min\left(R\,,\ \frac{\rho}{\nabla\rho}\right) \qquad (3)$$

In an isothermal sphere $b_{\max}$ is a factor of 2 smaller than the distance to the centre. For point like objects like SMBHs the effective minimum impact parameter a_{90} is given by the value for a 90° deflection using as typical relative velocity $v_{\rm BH}$,

$$a_{90} = \frac{GM_{\rm BH}}{v_{\rm BH}^2} \qquad (4)$$

which is for $X = 1$ a factor of two smaller than the gravitational influence radius. For extended objects like star clusters a good measure for the minimum impact parameter $b_{\min}$ is the half mass radius $r_{\rm h}$. If a point mass motion is numerically derived by a code with a smoothing length or grid cell size ϵ, the half mass radius should be substituted by $q\epsilon$ with a scaling factor q. As SFP we use $q = 1$ for PP and Tree codes, but for a PM code with cell size ϵ we expect $q = 0.5$, because the mutual interaction in the 2-body encounters is suppressed only where the particles are in the same cell. The resulting Coulomb logarithm is

$$\ln\Lambda \;=\; \ln\sqrt{\frac{b_{\max}^2 + a_{90}^2}{b_{\min}^2 + a_{90}^2}} \approx \ln\sqrt{\frac{l^2}{r_{\rm h}^2 + \left(\frac{GM_{\rm BH}}{v_{\rm BH}^2}\right)^2}} \qquad (5)$$

$$\approx\; \ln\left(\frac{l}{r_{\rm h}}\right) \quad \text{extended obj.} \qquad (6)$$

$$\approx\; \ln\left(\frac{l}{\frac{GM_{\rm BH}}{v_{\rm BH}^2}}\right) = \ln\left(\frac{l}{R}\,\frac{M(R)}{M_{\rm BH}}\right) \quad \text{point like obj.} \qquad (7)$$

where the last equality holds for circular orbits in spherically symmetric systems ($M(R)$ is the enclosed back ground mass inside radius R). These approximations can be used as long as $\ln\Lambda > 1$.

2 ORBITAL DECAY IN GALACTIC CENTRES

Here we give explicitely the solution for a massive object moving on a circular orbit in a cuspy central region with a power law density profile given by the cumulative mass

$$M(y) = M_0 y^\alpha \quad \text{with} \quad y = \frac{R}{R_0} \qquad (8)$$

with an isotropic velocity distribution function. For convenience we normalise all quantities to the initial values. The initial position of the object is $R(t=0) = R_0$, the initial velocity is $v_{\rm BH}(t=0) = v_{\rm c0}$, the circular velocity at R_0. The normalised circular velocity $X_{\rm c} = \sqrt{2-\alpha}$ is constant during the orbital decay. With Eqs. 6 or 7 the Coulomb logarithm has the form

$$\ln\Lambda = \ln(\Lambda_0 y^\beta) \qquad (9)$$

with $\beta = 1, \alpha$ for extended and point like objects, resp.; $\beta = 0$ corresponds to the usual constant Coulomb logarithm. The orbital evolution is given by the implicit solution

$$t \;=\; \frac{\alpha+3}{\alpha+1}\frac{\ln z_0}{z_0}\left[\mathrm{Ei}(\ln z_0) - \mathrm{Ei}(\ln z(y))\right]\tau_{\rm df} \quad \text{with} \qquad (10)$$

$$\tau_{\rm df} \;=\; \left.\frac{-v_{\rm BH}}{\dot v_{\rm BH}}\right|_{\rm R_0} = \left[2\pi\alpha\chi\ln\Lambda_0\frac{M_{\rm BH}}{M_0}\right]^{-1}\tau_{\rm orb} \quad ; \qquad (11)$$

$$\tau_{\rm orb} \;=\; \frac{2\pi R_0}{v_{\rm c0}} \quad ; \quad z = \Lambda^{\frac{\alpha+3}{2\beta}} = z_0 y^{\frac{\alpha+3}{2}} \qquad (12)$$

For more details of the orbits $y(t)$ see App. A. In the limit of constant Coulomb logarithm we recover the explicite solution of SFP

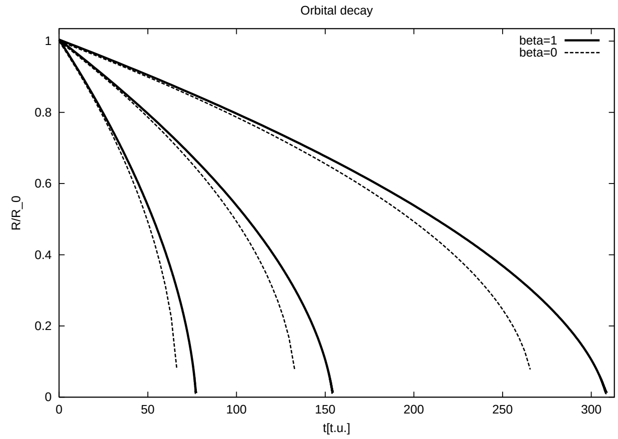

Figure 1. Orbital decay for 1, 2, , and 4 times the original mass $M_{\rm BH} = 6.23\,10^4 M_\odot$ from right to left. The time unit is $1\,t.u. = 0.14\tau_{\rm orb}$. Full lines are with $\ln\Lambda$ of Eq. 6 and dashed lines are the corresponding lines with constant Coulomb logarithm $\ln\Lambda = 4.7$ (as in SFP).

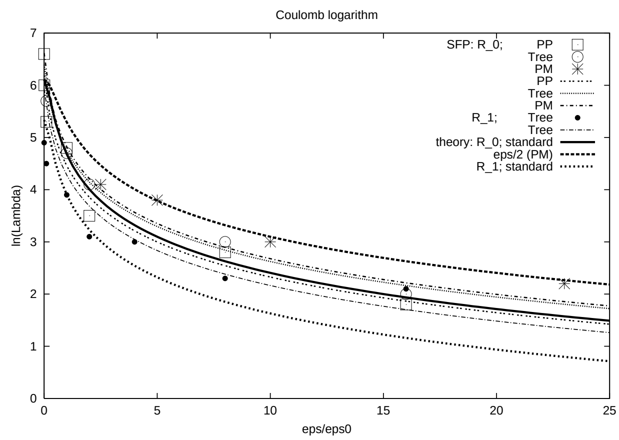

Figure 2. Here we show the numerically determined values for $\ln\Lambda$ and the corresponding analytical fitting functions for the different codes of SFP with starting position at R_0 and also the Tree code values starting at $R_1 = 0.46R_0$ with enclosed mass $M_1 = M_0/2$. The full and dotted line are our standard function (Eq. 5) with $b_{\min} = \epsilon$ for the PP and Tree code, and the dashed line corresponds to the PM code using $b_{\min} = \epsilon/2$.

$$y = \left[1 - \frac{\alpha+3}{\alpha+1}\frac{t}{\tau_{\rm df}}\right]^{\frac{2}{\alpha+3}} \qquad (13)$$

For a direct comparison with SFP we use $\alpha = 0.9$ and the initial values $R_0 = 7pc$, $M_0 = 5.9\,10^7 M_\odot$, $M_{\rm BH} = 6.23\,10^4 M_\odot$. The time unit in SFP is $1\,t.u. = 0.14\tau_{\rm orb}$. With these values we get for the standard case with $b_{\min} = \epsilon_0 = 0.03pc$ the result $\ln\Lambda_0 = 4.7$ (Eq. 6), which is exactly the best fit value for the tree code results of SFP. In Fig. 1 a direct comparison of the orbital evolution for different object masses with constant and linearly decreasing Λ is shown. In the later phase of evolution, the orbital decay is delayed by about 20%. But an inspection of the N-body data plotted in SFP shows, that from the comparison with single runs no conclusions at that level can be derived. More significant is the systematic behaviour with varying smoothing length ϵ. In Fig. 2 we replot the best fit values for $\ln\Lambda$ and the corresponding analytical fit functions of SFP and compare this with our theoretically based approximation for $\ln \Lambda_0$ (Eq. 5) without any fitting correction. We used for the PP and Tree code $b_{\min} = \epsilon$ and for the particle mesh code $b_{\min} = \epsilon/2$ as discussed in the introduction. Also plotted are the data, when starting at $R_1 = 0.46 R_0$, the radius with $M_1 = M_0/2$, where we expect an initial Coulomb logarithm $\ln \Lambda_1 = \ln \Lambda_0 - 0.78$. The systematic variation fit quite well when taking into account a noise of $\Delta \ln \Lambda = 0.3$ (as quoted by SFP).

3 CONCLUSION

For a general use we give a simple theoretical approximation (Eq. 5) to the Coulomb logarithm in the dynamical friction formula (Eq. 1) in order to generalise the usually used constant value. Using this new Coulomb logarithm with the local scale length l as maximum impact parameter and with the orbital velocity as a typical relative velocity for the lower cutoff a_{90} in the point mass case we find without any fitting correction a consistent value for the initial $\ln \Lambda_0$, the correct mass dependence, and the numerically observed transition from extended to point like objects of the independent investigation of SFP. The smoothing length ϵ of particle-particle or Tree codes can be used up to a factor of order unity as the minimum impact parameter. In contrast to SFP we confirm for the particle mesh code that one should use half the cell size as minimum impact parameter. It should be reminded however, that from the structure of $\ln \Lambda$, there is implicitly one common scaling factor for $b_{\max}, b_{\min}, a_{90}$ free. So the normalisation of one of these quantities must be fixed in order to determine the other two. This was done in Just & Peñarrubia (2002) for a_{90}. We also give an analytical formula for the evolution of circular orbits in power law density profiles with a position dependent Coulomb logarithm. The position dependence of $\ln \Lambda$ leads to a delay in the orbital decay of about 20% only, but for the evolution of eccentric orbits or for the application of a single formula in a large parameter space, the systematic parameter dependence of $\ln \Lambda$ will be significant.

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APPENDIX A: ORBITAL DECAY

Here we derive explicitely the orbital decay of a massive object on a circular orbit in a cuspy density distribution with a position dependent Coulomb logarithm. We adopt a power law profile with isotropic velocity distribution function for the background distribution and find

$$M(y) = M_0 y^{\alpha} \quad ; \quad v_{\rm c} = v_{\rm c0} y^{\frac{\alpha-1}{2}} \tag{A1}$$

$$\sigma = \frac{v_{\rm c}}{\sqrt{2(2-\alpha)}} \quad ; \quad \Lambda = \Lambda_0 y^{\beta} \tag{A2}$$

for the cumulative mass, the circular velocity, the velocity dispersion, and the ratio of impact parameters, respectively. All values are normalised to the values at the initial distance R_0. The distance to the centre is $y = R/R_0$. We have $\beta = 0,\ 1,\ \alpha$ for a constant Coulomb logarithm, an extended object (Eq. 6), and a point like object (Eq. 7), respectively. For simplicity and comparison with SFP we use a Gaussian velocity distribution function and get a constant value for χ, since $X_{\rm c} = \sqrt{2-\alpha}$ is independent of y. The orbital decay can be computed by identifying the angular momentum loss due to dynamical friction $\dot{v}_{\rm BH} R$ with the radial variation of angular momentum $\dot{R} {\rm d}(v_{\rm c} R)/{\rm d}R$. This leads to the normalised differential equation

$$\tau_{\rm df} \dot{z} = -\frac{\alpha+3}{\alpha+1} \frac{z_0}{\ln z_0} \ln z \tag{A3}$$

(for $\tau_{\rm df}$ and z see Eqs. 11 and 12, resp.). The solution can be given in implicit form (2.724.2 of Gradshteyn, Ryzhik (1980))

$$\frac{t}{\tau_{\rm df}} = \frac{\alpha+1}{\alpha+3} \frac{\ln z_0}{z_0} \left[{\rm Ei}(\ln z_0) - {\rm Ei}(\ln z(y)) \right] \tag{A4}$$

$$\approx \frac{\alpha+1}{\alpha+3} \left[1 + \frac{2\beta}{\alpha+3} \frac{1}{\ln \Lambda_0} - y^{\frac{\alpha+3}{2}} \frac{1 + \frac{2\beta}{\alpha+3} \frac{1}{\ln \Lambda_0 + \beta \ln y}}{1 + \beta \frac{\ln y}{\ln \Lambda_0}} \right]$$

with the exponential-integral function Ei and its asymptotic expansion (8.211.2 and 8.216 of Gradshteyn, Ryzhik (1980), resp.). The last approximation is useful for implementation and shows also that we recover the solution for constant Coulomb logarithm in the limit $\beta = 0$ (Eq. 13).

In Fig. A1 the orbital decay is plotted for different power low indices α for constant and varying $\ln \Lambda$ and for point like and extended objects. The initial values for the distance R_0, the enclosed mass M_0, the object mass $M_{\rm BH}$ are the same for all curves, but the initial Coulomb logarithm $\ln \Lambda_0$ and the density ρ_0 depend on α. The effective decay time $t_{\rm dc}$ is given by

$$t_{\rm dc} = \frac{\alpha+1}{2\pi\alpha(\alpha+3)} \frac{1}{\chi \ln \Lambda_0} \frac{M_0}{M_{\rm BH}} \left[1 + \frac{2\beta}{(\alpha+3)} \frac{1}{\ln \Lambda_0} \right] \tau_{\rm orb} \tag{A5}$$

where the factor in square brackets originate from the position dependence of $\ln \Lambda$.

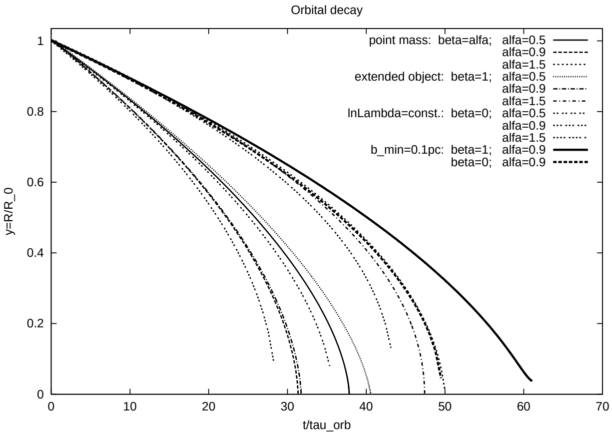

Figure A1. Orbital decay in cusps with $\alpha = 0.5, 0.9, 1.5$ using the initial values of SFP, i.e. $R_0 = 7pc$, $M_0 = 5.9\,10^7 M_\odot$, $M_{\rm BH} = 6.23\,10^4 M_\odot$. This leads to the initial orbital time $\tau_{\rm orb} = 0.224 Myr$ and $\ln\Lambda_0 = \ln\frac{950}{3-\alpha}$ for the point mass case, which is used for all values of β to be comparable. The last two lines show that the effect of a position dependent Coulomb logarithm is larger for more extended objects ($\ln\Lambda_0 = 3.5$).