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A stochastic Monte Carlo approach to model real star cluster evolution, II. Self-consistent models and primordial binaries

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ABSTRACT

The new approach outlined in Paper I (Spurzem & Giersz 1996) to follow the individual formation and evolution of binaries in an evolving, equal point-mass star cluster is extended for the self-consistent treatment of relaxation and close three- and four-body encounters for many binaries (typically a few percent of the initial number of stars in the cluster mass initially). The distribution of single stars is treated as a conducting gas sphere with a standard anisotropic gaseous model. A Monte Carlo technique is used to model the motion of binaries, their formation and subsequent hardening by close encounters, and their relaxation (dynamical friction) with single stars and other binaries. The results are a further approach towards a realistic model of globular clusters with primordial binaries without using special hardware. We present, as our main result, the self-consistent evolution of a cluster consisting of 300.000 equal point-mass stars, plus 30.000 equal mass binaries over several hundred half-mass relaxation times, well into the phase where most of the binaries have been dissolved and evacuated from the core. The cluster evolution is about three times slower than found by Gao et al. (1991). Other features are rather comparable. At every moment we are able to show the individual distribution of binaries in the cluster.

Key words: stellar dynamics – star clusters – numerical methods – binaries, primordial

1 INTRODUCTION

Dynamical modelling of globular clusters and other collisional stellar systems (like galactic nuclei, rich open clusters, rich galaxy clusters) still suffers from severe drawbacks. They are due partly to the poor understanding of the validity of assumptions used in statistical modelling based on the Fokker-Planck and other approximations on one hand, and due to statistical noise and the impossibility to directly model realistic particle numbers with the presently available hardware, on the other hand.

Only recently a detailed comparison of the different methods for comparable parameter choices has been tackled (Giersz & Heggie 1994a,b, henceforth GHI, GHII, Giersz & Spurzem 1994, henceforth GS, Spurzem & Aarseth 1996, Giersz 1996, 1998, Spurzem 1996). They include theoretical models as the direct numerical solution of the orbit-averaged Fokker-Planck equation for isotropic systems (Cohn 1980), its 2D generalization to anisotropic models (Takahashi 1995, 1996, 1997, Takahashi, Lee & Inagaki 1997) and rotating axisymmetric clusters (Einsel & Spurzem 1999), isotropic (Heggie 1984) and anisotropic gaseous models (Louis & Spurzem 1991, Spurzem 1994) and direct N-body simulations using standard N-body codes (NBODY5: Aarseth 1985, Spurzem & Aarseth 1996; NBODY4: Makino & Aarseth 1992, Makino 1996, Aarseth & Heggie 1998, NBODY6 and NBODY6++: Aarseth 1996, 1999, Spurzem 1999) or Monte Carlo schemes (Giersz 1996, 1998). All the cited work, however, only dealt with idealized single-mass models. There are very few attempts yet to extend the quantitative comparisons to more realistic star clusters containing different mass bins or even a continuous mass spectrum (Spurzem & Takahashi 1995, Giersz & Heggie 1997, Heggie et al. 1998).

The results could be summarized by saying that in general the Fokker-Planck approximation (small angle two-body scattering dominates the global evolution of the system), the approximation of heat conduction (its energy transport can be treated as heat conduction in a collisional gas), and the statistical binary treatment (model of energy generation by formation and subsequent hardening of three-body binaries using simple semi-analytical estimates) all appear to be a fairly good description of what happens in N-body simulations. But there are still two basic drawbacks: (i) all comparisons are so far limited to rather small particle numbers ($N \leq 64000$) as compared to real particle numbers of globular clusters of the order of a few 10^5 or even up to 10^6 stars. Low N models cannot be easily extrapolated to higher N, since after core collapse a variety of different processes (close encounters, tidal two-body encounters, effects of the finite size of the stars) all vary with time scales, which depend on different powers of the particle number (see e.g. the scaling problem tackled by Aarseth & Heggie 1998); (ii) during core bounce and binary driven post-collapse evolution an individual N-body simulation exhibits stochastic fluctuations, due to the stochastic occurrence of superelastic scatterings of very hard binaries with field stars and other binaries (three-body and four-body encounters, henceforth briefly "3b" and "4b" encounters). Although the averaged evolution of the system, understood either as a time average (looking for long post-collapse times) or as an ensemble average (averaging statistically independent single N-body models), is reproduced well by the theoretical models based on the above assumptions, the *individual* evolution of a stellar system, even with a relatively large particle number, might not be exactly matched at any instant. The most recent collaborative experiment in this area (Heggie et al. 1998) gives a good overview: all methods do agree fairly well, but variations of quantitative results of some 10 or 20 % and some scaling problems, which are not exhaustively examined, have to be tolerated.

Such considerations led to the construction of special-purpose computers for direct N-body simulations (Sugimoto et al. 1990, Makino et al. 1997, Makino & Taiji 1998) and considerable efforts to improve the highly accurate N-body simulation software used (see e.g. Aarseth 1996, 1999, Spurzem 1999, but also the alternative approach by KIRA mentioned e.g. in Makino & Taiji 1998, McMillan & Hut 1996, Portegies Zwart et al. 1998). But there is an elegant alternative way to generate models of star clusters, which correctly reproduce the stochastic features of real star clusters, but without really integrating all orbits directly as in an N-body simulation. These so-called Monte Carlo models were first presented by Hénon (1971, 1975, Spitzer 1975) and later improved by Stodółkiewicz (1982, 1985, 1986) and in further work by Giersz (1996, 1998), Joshi, Rasio & Portegies Zwart (2000), Joshi, Nave & Rasio (2000). The basic idea is to have pseudo-particles, whose orbital parameters are given in a smooth, self-consistent potential. However, their orbital motion is not explicitly followed; to model interactions with other particles like two-body relaxation by distant encounters or strong interactions between binaries and field stars, a position of the particle in its orbit and further free parameters of the individual encounter are picked from an appropriate distribution by using random numbers. For a more detailed description see the cited papers and Sect. 2 below.

In the past it proved to be very delicate to properly tailor flexible and versatile pure Monte Carlo models. Stochastic fluctuations of the parameters of N-body systems with large particle numbers (say $N \geq 10000$) are mainly due to binary effects (formation by 3b encounters, superelastic scatterings with field stars, or, in the case of the presence of many primordial binaries, binary-binary encounters, see e.g. Heggie & Aarseth 1992, henceforth HA92), not so much due to the inherent fluctuations resulting from the discrete nature of the particle system (which should decay approximately with $\sqrt{N}$). Therefore the idea appeared only to model the binary population by a Monte Carlo technique, above a background of single stars, which are treated by a standard theoretical model. Takahashi & Inagaki (1991) published a similar approach for the case of an isotropic Fokker-Planck model, but without following individual binary's orbits and their relaxation interaction with themselves and other single stars. In an earlier approach Inagaki & Hut (1988) simulated the stochastic evolution of a binary population including a distribution of binary orbits. However, their background single star cluster was only approximately modelled, and their binaries were assumed to have purely radial orbits, and dynamical friction was treated approximately (as opposed to a full Monte Carlo model, which models explicitly the cumulative effect of many small angle encounters in order to generate proper relaxation and dynamical friction effects).

Here we want to extend the stochastic treatment of binaries in an anisotropic gaseous model for the single stars, as introduced in Spurzem & Giersz (1996, henceforth Paper I), for the self-consistent inclusion of many primordial binaries in a globular cluster model (which is yet a simplified model, with regard to its neglect of any finite size effect of the stars and its assumption that all stars have equal masses). It is very clear now from the observational (Hut et al. 1992) as well as the theoretical viewpoint of star formation that globular clusters do contain of the order of 5-15 % of binaries, which means that at the time of their formation the binary fraction had to be even larger, because many soft binaries are subject to disruption by close 3b and 4b encounters. Nevertheless theoretical modelling of the dynamical evolution of even idealized large star clusters containing many binaries has not advanced very much in the last years.

Pioneering studies by McMillan, Hut & Makino (1990, 1991), McMillan & Hut (1994) using a direct N-body integrator with regularization techniques for close binaries (NBODY5: Aarseth 1985) used some 1000 particles of equal mass only (binary membership of the order of 10-20 %). A similar work by HA92 provided in some more detail went up to $N = 2000$ and later up to $N = 6000$ Aarseth & Heggie (1993), with less than 10 % of binaries. The results of such models was that the details of the initial distribution of primordial binaries, at least for such small total particle numbers, strongly influences the global dynamics of the star cluster, which is quite understandable, since usually the binaries' total binding energy is much larger than the binding energy of the entire star cluster.

On the high-N branch Fokker-Planck modellers tried to incorporate appropriate average cross-section for 3b and 4b encounters in their simulations, as was done so successfully for the 3b case (Lee et al. 1991, Giersz & Spurzem 1994). In a pioneering study Gao et al. (1991, henceforth GGCM91) published the first self-consistent model of an N=330.000 star cluster with as many as 30.000 binaries and showed how gravothermal collapse (and subsequent oscillations) were delayed by a large factor in time due to previous "burning" of primordial binaries. In their model the close binary encounters, however, had to be treated in a very approximate way, as we will discuss later.

Unfortunately the direct N-body modelling of a case like the one in the GGCM91 paper is hardly possible today even with the fastest special purpose computers, because both the scaling of the computational time for the general N-body problem is prohibitive and as well as the regularization of the close binaries downgrades the performance very much. In this paper we use the hybrid Monte Carlo code presented in Paper I, which combines an anisotropic gaseous model for the single star component with a Monte Carlo stochastic treatment of many individual binaries. In our largest models we are able to follow 300.000 stars and 30.000 binaries as in the paper by GGCM91. Our model is entirely self-consistent in the way that both the binaries and single stars are subject to their own and the other component's gravitational field. The binaries are treated individually with their orbital and internal parameters (eccentricity, binding energy, semi-major axis). Relaxation of the binaries with themselves and the single stars as well as close 3b and 4b encounters are treated in a direct, but stochastic way using approximate cross sections for the latter and the standard Monte Carlo procedure (Hénon 1971, Stodółkiewicz 1982, 1986, Giersz 1998) for the relaxation. We use the same probability distribution for the hardening of the harder binary by close 4b encounters as GGCM91 (originating from Mikkola 1983a, b, 1984a, b); different to them, and more realistically, we compute the probability for a close 4b encounter to occur in a self-consistent way from the local density distribution of our binaries. We can also in detail follow the evolution and compile balances of all the reaction products in any close encounter (singles and binaries).

At this place some more sophisticated models of the evolution of a large binary population should not go unmentioned, i.e. the Hut et al. (1992) model with a very detailed study of finite-size effects and using unequal masses, similarly Sigurdsson & Phinney (1995), but both using a static background of single stars. Another interesting subject related to this study is the question of the evolution of binary parameters in young star forming clusters in the galaxy, which was studied by Kroupa (1995) by using NBODY5 for direct N-body simulations. Our model will be soon further developed to incorporate multi-mass systems, which will render it applicable for all such purposes.

In the next section we will describe only those features of the model which have been added as compared to Paper I. For the treatment of the ordinary relaxation process and the close 3b encounters we refer to Sect. 2 of Paper I.

2 SELF-CONSISTENT MONTE CARLO TREATMENT OF BINARIES

2.1 General remarks

In Paper I stochastic binaries were followed in their motion and interactions (two-body relaxation or dynamical friction, and close 3b encounters) with the single stars. The only feedback to the single stars, which were treated by the gaseous model, was the kick heating due to superelastic 3b scatterings. In order to improve our models to make them applicable for systems with many (primordial) binaries one has to cope with a number of complications. First, the energy balance is changed due to 4b (binary-binary) close encounters, and binary-binary small angle relaxation has to be taken into account according to Hénon's Monte Carlo method (Hénon 1971). Second, the energetic feedback effect of binary-single star relaxation cannot be neglected any more for the energy balance of the single stars. Third, binaries move in the system due to relaxation and close encounters, and so they change the total mass and potential of the entire system. Another complication occurs due to the losses and gains of single stars by binary formation and disruption of binaries in close 4b encounters. All such adjustments generate a feedback via the potential gradient in Euler's equation to the single stars. The first points were relatively easy to implement: we adopted the approximate 4b cross sections given by GGCM91 (1991) as a working model, to be tested and improved later; the treatment of binary-binary relaxation is similar to Giersz (1998), except that the local binary density is estimated here from the smoothed out binary mass (see below) via Poisson's equation. The second and third point, however, appeared to be extremely difficult to cope with. The problems were due to the very different nature of the gaseous and Monte Carlo treatment for the two components. The gaseous model approximates density and potential by smooth functions using Newton's method to find the iterative solution at the next time step. In contrast to this the stochastic binaries may experience rather quick and unsteady changes of their positions, velocities, and binding energies. Incorporating such changes in a naive way into the gaseous model (just adding up energy and mass balances per time step and applying them as a local source or sink term in the continuity and energy equations) yielded catastrophic failure. Thus we had to invent a number of measures to soften the transitions and links between the two systems without spoiling the physically correct treatment of the combined system and still preserve the advantages of the hybrid method. The physical justification is that the gaseous model is to be interpreted as an ensemble average over the evolution of many independent individual star clusters. So, when applying mass or energy changes we have to properly average out stochastic variations of the binaries. Our resulting code is technically fine-tuned and, frankly speaking, rather fragile according to our experience. However, with this paper we want to publish results obtained with a final code version, which we believe to produce physically correct results, in an efficient way, and without any unexplained or unphysical procedures. Due to difficulties described above we think it is justified to give in the following subsections some rather technical information how the implementation is done.

2.2 Binary mass distribution, relaxation, and heating processes

Our main idea to cope with the mentioned problems is to include the mass of each binary in the gaseous model as a smooth orbital mass distribution, computed by using the orbit parameters of the binary, which are known from the Monte Carlo model. For the stochastic treatment, the position of a binary is still determined by the appropriate picking procedure (Giersz 1998), and if we plot our results we give such positions of binaries. However, in Euler's equation in the gaseous model a smoothed out mass is used. Thus local potential fluctuations are reduced, since they now only reflect the change of the orbital parameters of the binary, not the stochastically picked individual binary position. So from the viewpoint of the gaseous model a binary appears as a smooth mass distribution, extending from its minimum to maximum orbital radius in the cluster. The mass distribution in the orbit is weighted according to dr/v_r. Thus the total binary mass $M_b(r)$ contained in a sphere of radius r, which enters into the force equation of the gaseous model is obtained from

$$M_b(r) = \sum_j M_{bj}(r) \tag{1}$$

where j is an arbitrary index counting the binaries and $M_{bj}(r)$ is computed by integrating the following equations:

$$\begin{aligned} M_{bj}(r) &= 0 \quad \text{for} \quad r < r_{\min}(E_j, J_j) \\ M_{bj}(r) &= m_{bj} \quad \text{for} \quad r > r_{\max}(E_j, J_j) \end{aligned} \tag{2}$$

with the total mass of the j-th binary m_{bj}; for $r_{\min} < r < r_{\max}$ we use

$$M_{bj}(r) = \frac{m_{bj}}{P_j} \int_{r_{\min}}^{r} \frac{dr}{v_r} \tag{3}$$

with the orbital period

$$P_j = \int_{r_{\min}}^{r_{\max}} \frac{dr}{v_r} \, . \tag{4}$$

Note that

$$v_r^2 = \frac{2}{m_{bj}}\big(E_j - \Phi(r)\big) - \frac{J_j^2}{m_{bj} r^2} \tag{5}$$

depends itself on r. The integrals in Eqs. 3 and 4 are solved by using Hénon's Monte Carlo procedure (Hénon 1971) to pick between 50 to 500 radial positions on the orbit (depending on how eccentric it is), compute v_r for each radius, and approximate the integrals in Eqs. 3 and 4 by a summation over these stochastically picked radii. The gravitational potential of the binary component is then computed in a standard way from the mass distribution using the approximation of spherical symmetry ($\Phi(r) \to 0$ for $r \to \infty$). A slight inconsistency enters here, because the new potential is not yet known in Eq. 5 for the computation of the new binary mass distribution (see Giersz 1998).

One more step is necessary to make the gaseous model compatible with many stochastic binaries. If its time step in high density phases becomes as small as some fraction of the central relaxation time, close encounters and relaxation encounters with sufficiently big deflection angles cause substantial heating and cooling in the stellar core. Note that the Monte Carlo model simulates the cumulative effect of small angle gravitative encounters by representative encounters, whose deflection angle does not need to be very small. Applying the energetic effect of such encounters immediately in the gaseous model component for the single stars would disrupt the stability of the system, because the time scale of the changes becomes much shorter than the local relaxation time. Therefore we define a reservoir of energy, which is applied to the single stars with a maximum rate of $E_c/t_{\rm rx,c}$, where E_c, $t_{\rm rx,c}$ are the core energy and the central relaxation time, which is smoothly distributed in the inner 20% of the total mass of the cluster, with a smooth power-law cutoff. We tested a number of other prescriptions (such as using the core or half-mass radius for that limitation), but it turned out that the 20% total mass cutoff provides the best compromise for a good description of the non-local energy input due to the binary heating. We interpret this as a simplified model of how the sub- or suprathermal reaction products themselves relax with the other cluster members. For the sink and source terms of the mass we use a similar procedure. With these measures the gaseous model code is able to cope with the induced mass and energy variations of an arbitrary number of binaries.

Heating terms contain contributions from the following processes:

- energy feedback of relaxation encounters between singles and binaries to the single stars. It is easily accomplished, since for each relaxation event in the standard Monte Carlo procedure we know the deflection angle and the energies of the reaction products after the encounter. It is checked whether the single star can escape as a result of its relaxation encounter with the binary. If this is not the case the energy it gained is added as a heating (or cooling) source to the gaseous model equations. The relaxation procedure itself is the same as described in Paper I, but now the energy feedback to single stars is not considered negligible.
- heating due to close 3b encounters. This is the classical heating by three-body binaries, which was first implemented in a steady approximation by Heggie (1984) and Bettwieser & Sugimoto (1984). Its Monte Carlo realization depends on an assumed probability for an encounter to occur and a probability distribution of ΔE_b, the change of binary binding energy (see Spitzer 1987, and detailed description in Paper I). Again, the heating is only applied if the single star reaction product of a close encounter does not escape. The treatment of this process is equal to Paper I.
- heating due to close 4b encounters. With many binaries in the system, the probability of close binary-binary encounters cannot be neglected any more. The problem is very difficult to solve. However, as in the case of 3b encounters, for hard binaries a typical event may be defined (at least in the case where all single stars have equal mass). The typical close encounter between two hard binaries generates a "hardening" (gain in binding energy) of the harder binary of the two and a disruption of the other one into single stars. Depending on the initial binding energies the single stars created may escape. If not they contribute to the heating term described above. In detail we follow closely the procedure and cross sections outlines by GGCM91, because one of our main aims is to compare our models with their work. To determine the probability for a close 4b encounter we sort all binaries radially according to their most recently picked position in their orbits and group them into pairs. For each pair the probability is computed and compared to a random number, insofar this is again the standard Monte Carlo treatment (Giersz 1998). The probability P_{bb} for a close encounter to occur is determined, however, here by using an $n\sigma v$-ansatz:

$$P_{bb} = n_b \pi R_b^2 v_{\rm rel} \cdot \left(1 + \frac{2Gm_b}{R_b v_{\rm rel}^2}\right) \tag{6}$$

where $n_b = \rho_b/m_b$ is the local binary number density (determined from the smoothed out binary mass via a spherically symmetric Poisson's equation), m_b the binary mass, $v_{\rm vel}$ the relative velocity of the two selected binaries, and R_b is 2.5 times the semi-major axis of the harder binary (Stodółkiewicz 1986). In his paper a different ansatz was given for P_{bb}, but we checked that by order of magnitude it provides the same results. By comparing P_{bb} with a random number it is decided whether an interaction takes place. If so, we use the probability distribution given by GGCM91 (taken from earlier work by Mikkola 1983 a, b, 1984 a, b)

$$G(y) = \frac{49}{4} y \left(1 + \frac{7}{2} y^2\right)^{-11/4} \tag{7}$$

which gives the fractional translational or recoil energy E_r released in the encounter, defined by

$$y = \frac{E_r}{E_b + E_b'} \tag{8}$$

(E_b, E_b' being the two binding energies). The increase in binding energy of the harder binary is thus

$$\Delta E = y(E_b + E_b') + E_b' \tag{9}$$

since the softer binary has to be disrupted. One quarter of the recoil energy E_r is assumed to be carried off by the remaining binary, three quarter are distributed randomly between the two stars. Such a procedure is very approximate and it neglects certain other channels of binary-binary reactions, such as the formation of stable hierarchies with three or more particles (compare Aarseth 1999). While our Monte Carlo model in future work will be capable to include more details of 4b encounters, for the model presented here we decide to use the same reaction processes as GGCM91, for a comparison of our results with theirs. We use the Monte Carlo rejection technique to pick random values from a distribution $f(z) = z^{3/4}$; in case a value is accepted we determine

$$y = \sqrt{\frac{2}{7}\frac{1-z}{z}} \tag{10}$$

where z is the original random number. This provides the above probability distribution $G(y)$. By checking the translational energies of the reaction products (one binary and two singles) we determine whether they escape or not. If the single stars do not escape, their translational energy is added to the heating source for the single star component, and their mass is added to the single stars as well. Again, as described above for the energy, the mass is taken away from or added to the single star component smoothly in a region defined by the 20% Lagragian radius with a power-law cutoff.

2.3 Adjusting the masses and energies

The following processes change the masses contained in the single or binary components and are properly accumulated and taken from or given to the single stars in form of a source or sink term in the continuity equation. Note, that direct and indirect heating processes are taken automatically into account as the outcome of close 3b and 4b encounters. In the following we focus on how that is related to the mass transfer between the components and mass loss of the system.

- formation of three-body binaries; this process is treated as in Paper I, but now the mass loss for the single stars is properly accounted for as a sink term in the gaseous model equations;
- generation of two single stars as a reaction product from a 4b encounter, which do not escape; this is accounted for in a mass source term in the continuity equation;
- mass loss of singles and binaries by relaxation, close 3b and 4b encounters. This is mass loss to the entire system which causes its total binding energy to decrease.
- finally, there is an ongoing motion of binaries in the system, due to relaxation with each other and with single stars (dynamical friction) and due to 3b and 4b encounter kicks.

To check total energy conservation we use a balance equation with respect to the total energy $E_{\rm tot}$ of the single stars only:

$$2E_{\rm tot} = \mu v^2 + \mu(\Phi_s + 2\Phi_b) \tag{11}$$

where Φ_s, Φ_b are the local potentials from singles and binaries, respectively, and v is the 3D velocity dispersion of the single stars. Note, that the factor of 2 in front of the binary potential here occurs, because the binaries provide an external potential and do not belong to the self-gravitating single star system. To simplify the following discussion we consider the above as an energy balance for one discrete radial shell in our model. To get the total energy this has to be integrated (or in the discrete numerical model summed) over the volume. In that sense $\mu = \rho \Delta V$ is the mass of single stars contained in one radial shell. Any changes in our model cause a change

$$2\delta E_{\rm tot} = v^2\delta\mu + \delta\mu(\Phi_s + 2\Phi_b) + \mu\delta v^2 + \mu(\delta\Phi_s + 2\delta\Phi_b). \tag{12}$$

Checking the total energy of the single stars in our model means book-keeping of all processes which create changes in one of the quantities of the above equation.

- first, there is a change of Φ_b due to relaxation, close 3b and 4b kicks of the binaries, binary escape, and escape of single stars which is caused by a binary-binary encounter. All such processes lead to changes in the orbital mass distribution of the binaries, but not to direct changes of the state of the single stars; thus we have

$$2\delta E_{\rm tot} = 2\mu\delta\Phi_b. \tag{13}$$

If however, single stars escape from the system which belonged as single stars to the cluster before (this is only possible by close 3b encounters between a binary and a single star) we have an energy change of

$$2\delta E_{\rm tot} = 2\mu\delta\Phi_s. \tag{14}$$

because the adjustment of the self-gravitating system proceeds such that $\Phi_s\delta\mu = \mu\delta\Phi_s$.

- second, there is mass loss of single stars due to recoil ejections in close 3b encounters. From our numerical experiments we find that in the case of such an adjustment the system reacts with $\delta v^2 \ll 1$ and $\mu\delta\Phi_b = -\mu\delta\Phi_s$ i.e. the thermal structure is not changed, and the change in the mass distribution is fully represented by the change of potential energy. Neglecting the first term and using the second identity in Eq. 12 yields:

$$2\delta E_{\rm tot} = v^2\delta\mu + \delta\mu(\Phi_s + 2\Phi_b) + \mu\delta\Phi_b. \quad (15)$$

• finally, there are two processes, which shift mass between singles and binaries, but do not change the total mass distribution, i.e. $\Phi_s + 2\Phi_b$ remains constant. They are formation of binaries by 3b encounters and disruption of binaries by 4b encounters leading to two non-escaping single stars. Here again the above equation seems to be the proper ansatz. However, we observe that in such a case the adjustment of the gaseous model proceeds in a way that

$$2\delta E_{\rm tot} = v^2\delta\mu + \delta\mu(\Phi_s + 2\Phi_b) + \frac{4}{3}\mu\delta\Phi_b. \quad (16)$$

This follows from the fact that the mass given to or taken from the single stars is not localized but distributed smoothly inside the central 20% of total mass, which introduces an additional shift in both the binaries' and singles' potential. Again from numerical experiments we find a good approximation of the identity $\delta\Phi_s/2 = -\delta\Phi_b/3$. We would like to stress that coefficients in front of the last terms on the right hand side of equations 15 and 16 are empirical and with good approximation equal to 1 and 4/3, respectively.

The reader should keep in mind, however, that all the described rather complicated details about the energy balance for the different processes do only matter for the purpose of checking the conservation of energy in the entire calculation. They have no influence on the physical events and progress of the model itself. But they were essential for us in the process of debugging the method and finding proper descriptions for all processes. In one of our models (Gao's run) the total binding energy of the binaries exceeds the binding energy of the system by several orders of magnitude, and this again is some orders of magnitude larger than the total energy error we find after a very long integration time. This is only possible if our procedure is correct.

2.4 Final remarks

Having described the zoo of different processes and balances in the previous subsections should illustrate to the reader, that in a stochastic binary model like ours we have a detailed and individual book-keeping of all these processes. It ensures that we have practically the same amount of information as in a large N-body simulation with primordial binaries. The underlying additional approximations here are just spherical symmetry, no relaxation processes other than standard two-body, and the cross sections for close 3b and 4b encounters. The latter will be improved to a detailed numerical three- and four-body integration in the future. N-body simulations with primordial binaries have been published by McMillan, Hut & Makino (1990), HA92, and Aarseth & Heggie (1993). In no case was the binary number larger than 180, and the total star number did not exceed 6000. We use the models of HA92 as a template for our models, to check whether there are differences and how significant they are. They (around 1992) needed 2000 h CPU time on a 10 Mflop workstation, hence the problem requires 72 Tflop. So it is a major computational job still today, though not a question of months. But our real interest stems from globular clusters, which may start with initial data like the ones used in the Fokker-Planck models of GGCM91, using 300.000 single stars and 30.000 primordial binaries.

It is questionable, whether this job can be done in a direct simulation even by using a Petaflops computer. We see the justification in our efforts to create stochastic and Monte Carlo models of star clusters with many binaries in the fact, that our model is with a computational effort of two weeks on a Pentium II PC, able to provide a full model for the GGCM91 case. In excess it yields a wealth of more details of all binary related processes and the full stochasticity of the binaries *as it could only be deduced otherwise from an enormous N-body simulation.* Note that some of the approximation used in our model will be released in the near future without fundamental difficulties, such as the cross sections for 3b and 4b interactions. On the contrary there are assumptions, which are rather fundamental for our model, and not straightforward to remove. They are spherical symmetry and the assumption that only two-body relaxation by small angle encounters is important. Such a treatment of relaxation may be justified, because collective processes are less important in large N-body systems. Regarding the assumption of spherical symmetry, all Monte Carlo models so far employed it, because it makes the potential computations much quicker and it always leads to purely deterministic orbits. This assumption is a much more severe restriction in the moment, because a treatment of axisymmetric systems would require a not straightforward reformulation of all Monte Carlo methods (similar as in the case of Fokker-Planck models, Einsel & Spurzem 1999).

One final note: the reader making comparisons between the results presented in the following sections as well as results by HA92, Hut, McMillan & Romani (1992), and GGCM91 should keep in mind that all results (except for the Fokker-Planck models of GGCM91) are individual representation of a statistical ensemble of solutions; all energy generation events for example occur randomly and the outcome of such close 3b and 4b encounters is treated here purely stochastic. So one should not expect a perfect match between all models, but an agreement of global and averaged parameters of the simulated systems. By using different random number sequences for our models, however, one can generate a statistical spread of the results, which is comparable to the differences between Monte Carlo models here and direct N-body results in HA92.

In the following we present different set's of test models by applying our new method, which we call Monte Carlo runs and Heggie's runs, due to the aim to compare with corresponding other works. Finally, we model a star cluster with 300.000 single stars and 30.000 binaries, called Gao's runs, which has no direct counterpart in the direct simulations, so we have to rely on the capabilities of our code and the comparison with the more approximate GGCM91 models.

3 RESULTS AND MODELS

3.1 Method

The dynamical equations of the gaseous model were discretized on a logarithmically equidistant mesh of 200 points ranging over eight orders of magnitude and solved by an implicit Henyey-Newton-Raphson scheme (see GS). More details on the Monte Carlo description of our binaries can be

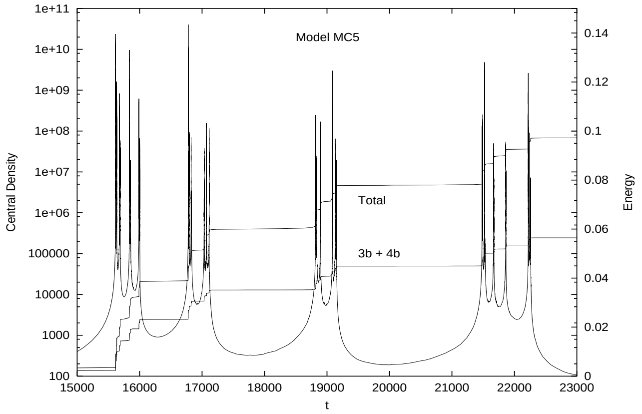

Figure 1. Model MC5, central density, total heating of the system, and heating due to 3b and 4b close encounters only, as a function of time.

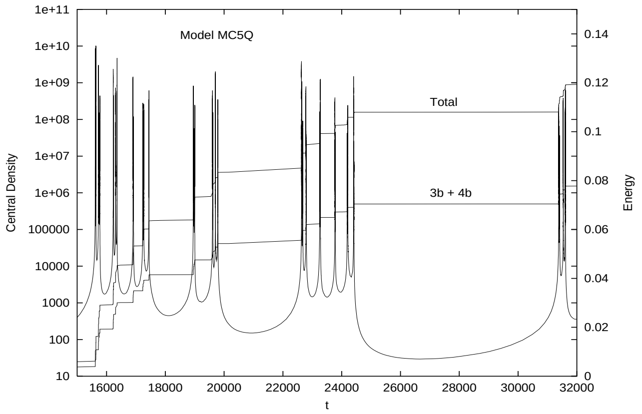

Figure 2. Model MC5Q, as Fig. 1

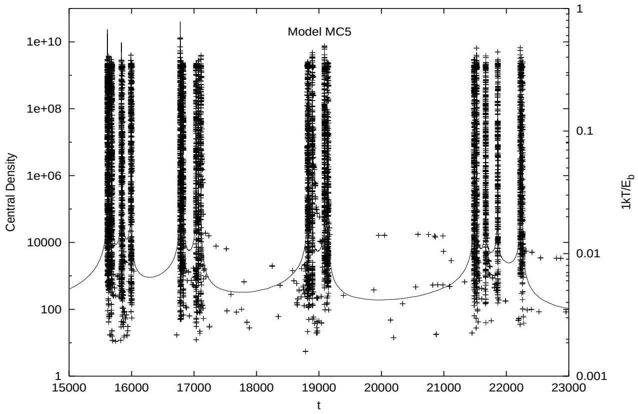

Figure 3. Model MC5, central density plus crosses for each close 3b and 4b encounter as a function of time. The abszissa of the crosses is chosen according to $1kT/E_b$, in order to illustrate the correspondence between events and fluctuations in the density curve.

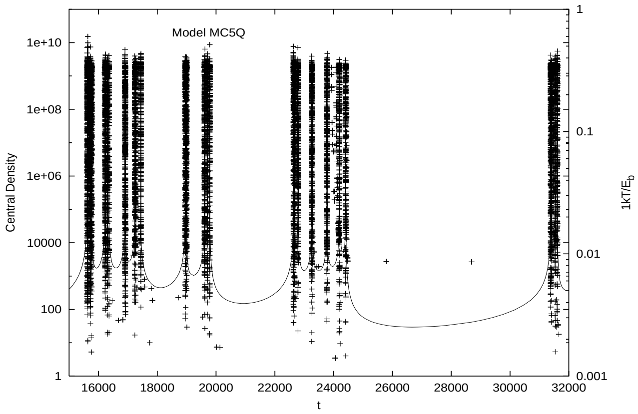

Figure 4. Model MC5Q, as Fig. 3

found in Paper I, the previous section and the cited literature. All results presented in the paper are (if not explicitly stated otherwise) given in standard units, i.e. $G = M = 1$ and $E = -1/4$ (Heggie & Mathieu 1986), where M and E denote the initial total mass and energy of the cluster (including single stars and binaries, but not the initial internal energy of the binaries).

3.2 Monte Carlo Runs

To check the reliability of the stochastic Monte Carlo model we studied models of a system consisting of 10^5 single stars without any primordial binaries. Binaries in these runs are created only due to the dynamical 3b interactions between single stars. In one run, labelled MC5, binaries only harden due to interactions with single field stars, but in another run, labelled MC5Q, they also interact between themselves. Both runs are compared to full Monte Carlo models discussed by Giersz (1998).

The evolution of the central density together with the different sources of heating of the gaseous system are presented in Figs. 1 and 2 for models MC5 and MC5Q, respectively. The curve labelled "3b + 4b" shows the heat transferred to the single stars (gaseous component) due to binary hardening (connected with their interactions with field single stars and other binaries). The curve labelled "Total" shows the total heating of the gaseous system including "3b + 4b" contributions (see above) plus an additional heating or cooling due to the motion of binaries in the system. Note that binaries from the point of view of the gaseous model form an external system. Their motion caused by small angle two-body interactions with single stars (relaxation process) induces changes of the total potential felt by the single stars. The total heating is always larger than that connected only with the strong binary interactions ("3b + 4b"), because the binaries preferentially lose their transational energy in the relaxation process and sink to the centre of the system. Therefore the relaxation interaction with the binaries is always, in an averaged sense, a heating (the "Total" curve is always above the pure "3b + 4b" curve). Note that individual relaxation encounters between binaries and single stars may lead to cooling also. This is in agreement with the expectation from the physical picture of mass segregation.

Large scale quasi-periodic oscillations are the most pronounced features of these two figures. They span several orders of magnitude in central density and have strongly varying periods. A characteristic feature of these oscillations is the lack of binary energy generation during the long phases of maximum expansion. It is widely accepted that this together with a temperature inversion (during these phases) are the most important signatures of gravothermal oscillations (Bettwieser & Sugimoto 1984, McMillan & Engle 1996).

In Figs. 3 and 4 crosses show the events of strong interactions between binaries and field stars and other binaries, together with the time evolution of the central density. It is clear that most of these interactions take place during the phases of maximum density, consistent with the picture of gravothermal oscillations. Here our runs also agree with results concerning the oscillations observed by Takahashi & Inagaki (1991) for a stochastic Fokker-Planck model (stochastic binary formation and energy generation of binaries, but no binary orbits and relaxation interactions with themselves and with single stars considered), by Makino (1996) for N-body simulations and by Giersz (1998) for full Monte Carlo simulations. Note that maxima of the central density observed in our model are a few orders of magnitude larger than in the full Monte Carlo model. This is connected with the way in which the central density is computed in the latter. It is estimated from the position and masses of a few innermost stars (for details see Giersz 1998), which can lead to an underestimation of the density. On the other hand, the high spatial resolution of the gaseous model in the core (where the innermost shell could consist of a mass of only a small fraction of a single star), and its extremely short time step during maximum collapse, both lead to very high central densities. It is worth mentioning that the refrigeration cycles (characteristic feature of gravothermal oscillations for continuum models, already shown in plots of the central velocity dispersion vs. central density in Bettwieser & Sugimoto 1984) are also present in our models (our plots of that phenomenon are not shown here in this paper). Though they are very noisy, at least four distinct cycles can be separated, which correspond to four different oscillation periods. Such periods occur from period doublings as they were studied when increasing N in detailed Fokker-Planck and gaseous models of gravothermal oscillations (Heggie & Ramamani 1989, Breeden et al. 1994, Spurzem 1994). The shape and the size of the loops in the refrigeration cycle are consistent with those in the pure Monte Carlo runs, though the latter showed even more fluctuations. It is remarkable, that the stochastic events occurring in both Monte Carlo models do not disturb much the path of the entire system in phase space, as it is provided by the continuum models.

In Figs. 5 for model MC5 and Figs. 6 for model MC5Q we provide snapshots of the binary distributions in energy-radius space of the cluster for four different time sets (see labels in the figures). The crosses shown in the figures represent all binaries which have interactions during the stated times (the number of crosses depend on the state of the system - large density means more interactions, small density less interactions). For model MC5 a bimodal distribution of binaries can be clearly seen - there are binaries with high binding energy far from the core and binaries in the vicinity of the core with a wide distribution of binding energies. In the course of evolution the build up of a reservoir of binaries in the outer halo (binaries in so–called parking orbits, as noted already by Hut, McMillan & Romani 1992) is clearly visible, and will be even more pronounced in models with many primordial binaries presented here in subsection 3.4.

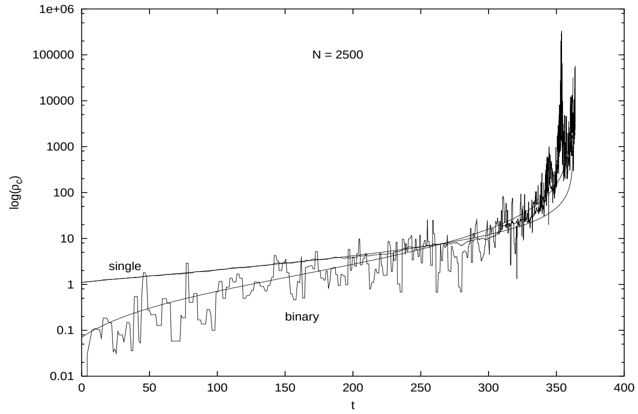

Figure 7. Comparison of two-component gaseous model with stochastic Monte Carlo binary model; central densities of single stars and binaries as a function of time, fluctuating curves belong to stochastic model, see further explanation in the text.

Such binaries have orbits extending very far out into the halo, and do not enter the core (where strong interactions preferentially take place). Binaries on parking orbits can be seen in the snapshot as vertically aligned sequences of cross symbols at the same constant binary binding energy. This occurs because in the Monte Carlo procedure the binaries in a nearly invariant orbit are repeatedly picked in different positions. Newly formed binaries appear in the bottom left corner of the figures and then in the course of evolution (interactions with single stars) they are migrating in the direction to the right and upwards. Finally they are ejected in a single strong interaction or put in a parking orbit in a less strong interaction. Note, the rather large number of binaries in parking orbits. In the full Monte Carlo models a much smaller number of such binaries is observed (only a few). The reason for this discrepancy is not clear, and may be connected with the already mentioned observation of higher central densities (leading to larger binary formation probability) in the maxima for our model as compared to pure Monte Carlo and N-body models. For model MC5Q the build up of the reservoir is also clearly visible, but with smaller number of binaries in parking orbits. Because of the presence of 4b encounters, which each destroy one binary, the number of binaries in the system drops faster, making the strip in the bottom of the figures less populated. Nevertheless, the bimodal distribution of binaries (parking orbits and core and its vicinity) is still well visible. The presence of strong 4b interactions in the system creates a wider distribution of binaries in energy (note that in each 4b interaction, while one binary is destroyed, the other is considerably hardened) as can be seen for example from comparison of the upper right plot of Fig. 5 and Fig. 6 for model MC5 and MC5Q, respectively.

In the next two subsections we will discuss the Monte Carlo stochastic models with many primordial binaries and compare their results with data available in the literature (HA92, GGCM91).

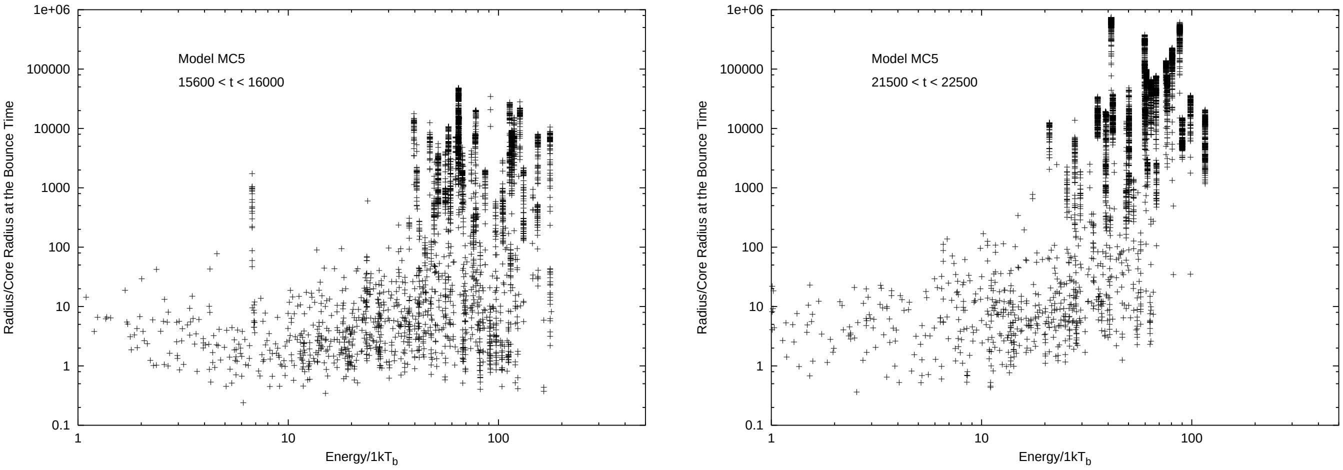

Figure 5. Model MC5, binary distribution in radius over core radius vs. binding energy in kT plane, for two different times as indicated in the plots; note the distinct sequences occurring due to the oscillations of the central density.

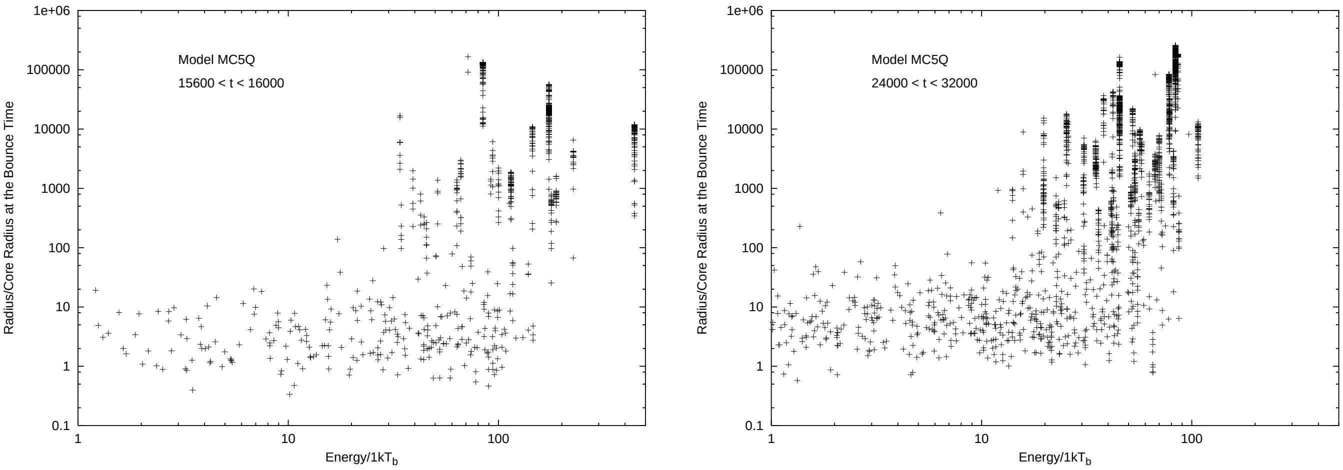

Figure 6. Model MC5Q, binary distribution in radius over core radius vs. binding energy in kT plane, for two different indicated times; note the distinct sequences occurring due to the oscillations of the central density.

3.3 Heggie's Runs

First, we have done one more test model in addition to those already discussed in Paper I, checking the correct mass segregation rate of a two-component system consisting of stars of masses $m_1 = 1$ and $m_2 = 2$. Hence the mass ratio is two, which mimics the situation of Heggie's model (HA92, see following Sect.) for which single stars and binaries are all composed of stars of the same mass. In Fig. 7 we show the central density evolution as a function of time for a two-component gaseous model (see Spurzem & Takahashi 1995, Spurzem 1992, the treatment is also equivalent to the same kind of gaseous model test runs presented in HA92) and for stochastic Monte Carlo model (using a gaseous model only for the single star component). To isolate the mass segregation effects only, all close 3b and 4b encounters were artificially suppressed in our model. As one can see there is an excellent agreement between both models over many orders of magnitude in the central binary density. The fluctuations of the binary density in the stochastic model are not present in the continuous gaseous model. There is also a few percent difference in the final core collapse time. It is of the same order as usual differences between collapse times of either different physical models (Fokker-Planck, N-body, gas), or even the scatter of collapse times between stochastic models themselves starting with different realizations of the initial model (using other random number sequence, see a detailed discussion of this effect in GHI).

Now we discuss a number of simulations which may be considered as test cases for our stochastic binary treatment in the refined version of the stochastic Monte Carlo method (as compared to Paper I). We denote these models by S, D, E, and SS, with the same naming as given in Table 1 of HA92. They are runs with 2500 stars each, containing 75 primordial binaries with binding energies in the range 2 to 20 kT (logarithmically equally distributed), where the spatial

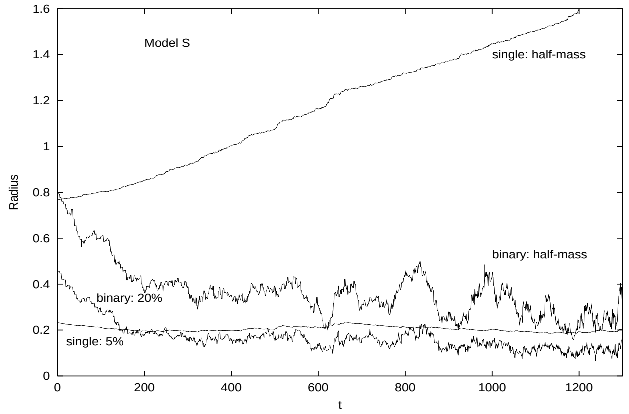

Figure 8. Model S, Lagrangian radii for the single stars and binaries (radii containing the innermost 5%, 50% of the single stars, and 20%, 50% of the bound binaries remaining in the cluster) as a function of time.

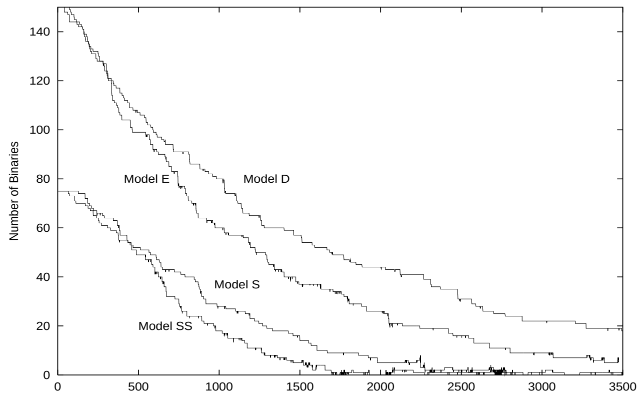

Figure 9. Model S, D, E, and SS; number of bound binaries as a function of time.

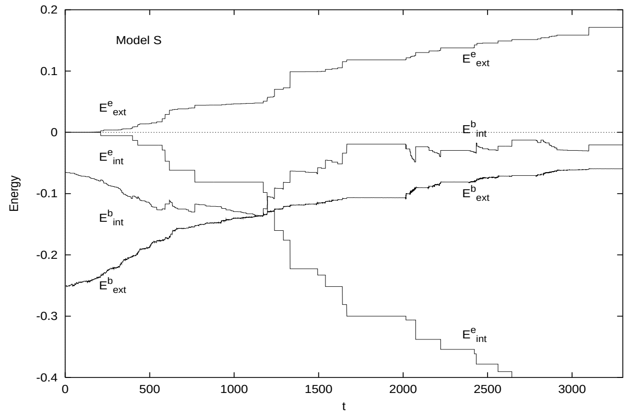

Figure 10. Model S, energy balances as a function of time in N-body units. The four different contributions are described in the text.

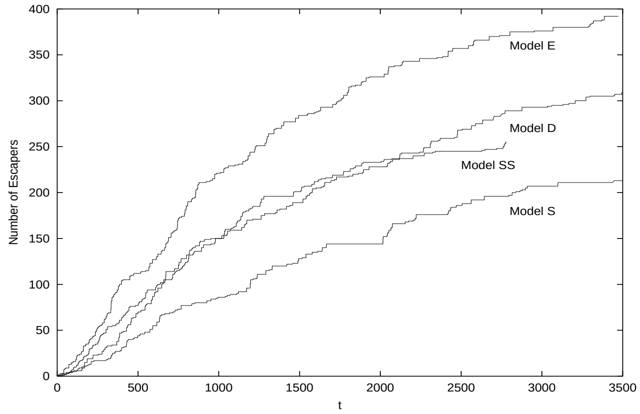

Figure 11. Model S, D, E, and SS; total number of escaping stars in singles and binaries as a function of time.

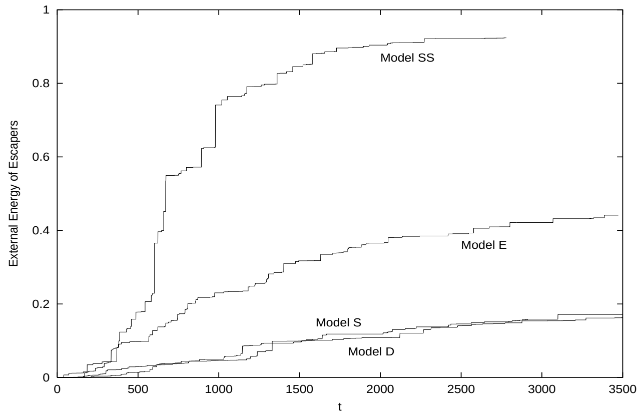

Figure 12. Model S, D, E, and SS; total external energy of escaping singles and binaries as a function of time.

distribution of the binaries is the same as that of the single stars (Plummer's model). While this describes the standard model S, there are model D (with double the number of binaries, but the same binding energy distribution), model E (double number of binaries, but stretched from 2 to 200 kT), and model SS (only 75 binaries as the standard model,

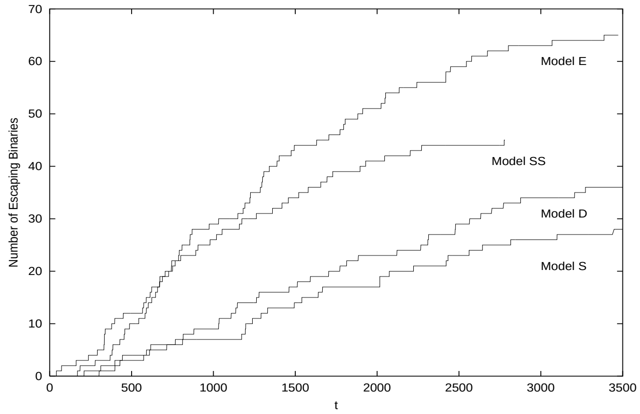

Figure 13. Model S, D, E, and SS; number of escaping binaries as a function of time.

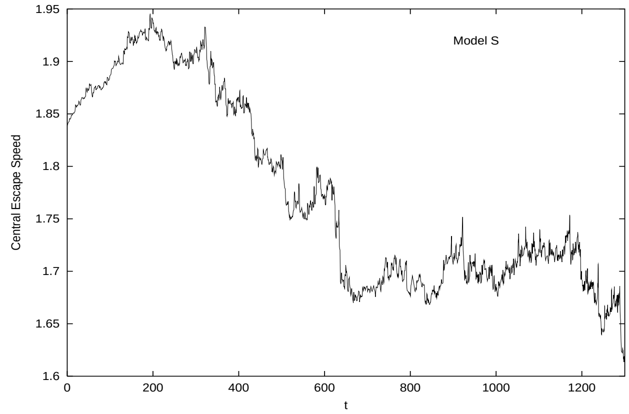

Figure 14. Model S, central escape speed as a function of time.

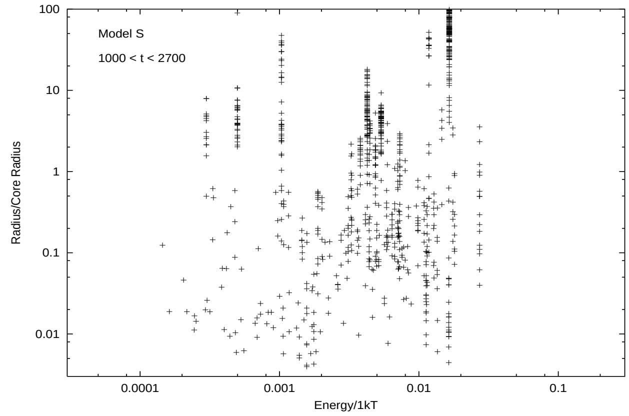

Figure 15. Model S, binary snapshots in a diagram plotting radius over initial core radius against binding energy in kT (initial), for the time $1000 < t < 2700$. There are more points than actual binaries because data of several time outputs have been plotted together.

but binding energy distribution extended from 2 to 2000 kT).

All our results match rather closely those of HA92, though each run takes only of the order of one hour on a Pentium II computer. We do not want to elaborate more on the physical interpretation, because this can be referred to HA92. Instead we present a number of figures in exactly the same manner as there, such that the reader can judge how well an agreement can be reached by our Monte Carlo type model. Clearly at such low particle numbers we cannot expect a complete match of the results. However, we have performed several runs for each model with varying random number seeds and our general conclusion is, that some features are stable while others exhibit stochastic variations. In our following arguments we will always try to discuss which differences to the N-body models we consider as "real" (in the sense that they may still point to deficiencies of our stochastic Monte Carlo model) and which we consider as a result of the statistical variations at small N.

Fig. 8 shows some Lagrangian radii separately for the single stars and binaries (see Fig. 8 of HA92). As in HA92 the segregation of binaries is very rapid up to about 100 time units. A more stationary phase follows, characterized

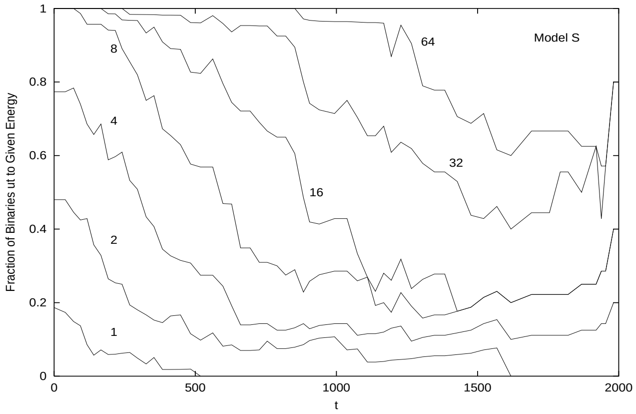

Figure 16. Model S, distribution of binding energies as a function of time for all bound binaries. The unit of energy is $2kT$, i.e. one and half times the initial mean kinetic energy of single stars. Each line gives the fraction of these binaries with binding energy less than the stated value.

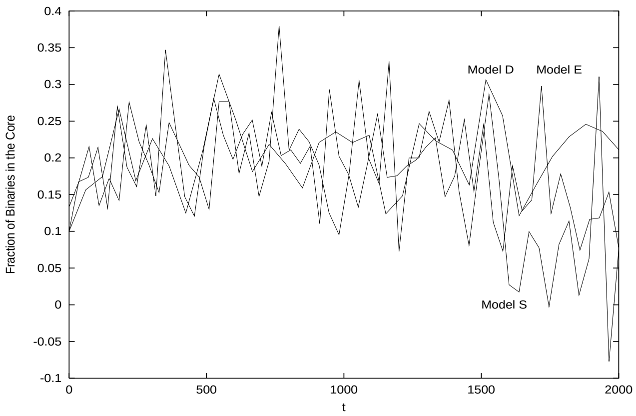

Figure 17. Models S, D, and E; fraction of binaries in the core as a function of time.

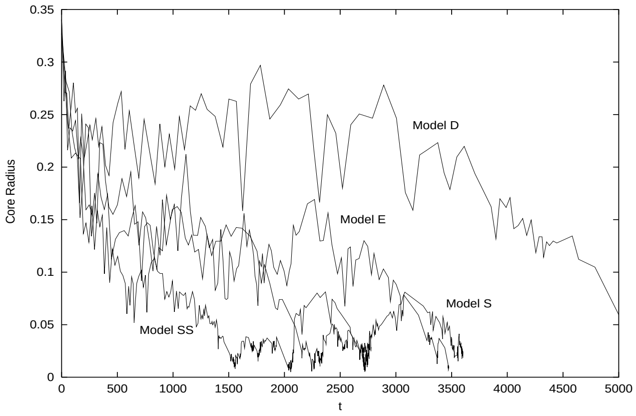

Figure 18. Models S, D, E and SS; core radius as a function of time.

by continuous binary destruction, and energy generation, until most of the binaries are destroyed. For a more detailed physical discussion see HA92. The somewhat faster growth of the half-mass radius in the gaseous model (as compared to the N-body result) is a typical feature resulting from different outer boundary conditions (see GHI).

Fig. 9 presents the number of binaries remaining bound in the system (see Fig. 9 of HA92). In contrast to HA92 after time $t \approx 1500$ (model E) or $t \approx 1000$ (model S) the steady binary destruction does not slow down, but continues in our models to rather small binary numbers. In HA92 thereafter the binary number remains larger. The variation of such plots with different random number initialization of the system has been explored. We found that it can explain to some extent the remaining differences, because stochastic variations are big in the late phases when there are only relatively few binaries left. The model we are using for our Fig. 9, for example, underwent core bounce relatively earlier than others, which explains the quicker destruction of binaries. Selecting another physically equivalent initial model can yield different collapse times, and very different features during the post-collapse phase (see GHI). Keeping in mind the different number of binaries at the late phases of evolution in our models and HA92 we want to stress, however, that the global energy budget for all models, here presented for our model S in Fig. 10 is in good agreement with the features seen in the direct N-body model (see Fig. 11 of HA92). For these plots we measure the total internal energy of the binaries remaining bound in the system ($E^{\rm b}_{\rm int}$) and of those binaries which escaped ($E^{\rm e}_{\rm int}$), and the total external (i.e. potential plus translational) energy of all objects (singles and binaries) remaining bound in the system ($E^{\rm b}_{\rm ext}$) and escaping ($E^{\rm e}_{\rm ext}$). Note, that $E^{\rm b}_{\rm ext}$ would be the standard total energy in a system without internal degrees of freedom (hence it starts at the canonical value of -0.25). As in HA92 we observe the initial phase of an increase of total external energy due to hardening of the binaries in the system (until $t \approx 1000$ in Fig, 10), which we would like to call the hardening phase, while afterwards total energy is gained in part by encounters leading to an escape of binaries with high binding energy. The total internal energy of bound binaries increases in this phase, which means that on average the typical binary in the system becomes less bound. We would like to call this the escaper phase. The reader is referred to a much more thorough discussion of the physical processes and balances determing these phases in HA92. Here we just want to stress that our model is able to reproduce all features of the N-body model.

Now we get to Fig. 11, where we have plotted the total number of escaping stars, where escaping binaries have been counted as two stars (see Fig. 12 of HA92). Compared to our results HA92 find much larger numbers, and no difference between model S and model D. They conclude that the presence of binaries does not enhance the rate of escape of single stars very much. In our isolated gaseous models we have not yet incorporated single star escapers due to relaxation and tidal fields, which is a subject of future work. This means that differences in the escape rates between our four models S, SS, D, E should be interpreted only in relation to the close 3b and 4b encounter processes. Hence our model reveals differences regarding the origin of single star escapers, which cannot easily be seen in the complete N-body of HA92, because their models are dominated by standard relaxation escapers. In contrast to the number of escapers, the external energy of escapers should in all models be dominated by the highly energetic 3b and 4b encounters. Consistent with this, our results for the external energy of all escaping objects (see Fig. 12 and Fig. 13 of HA92) agree much better with the HA92 model. The higher external energy of escapers for model E and SS can be attributed to the larger maximum binding energy of their binary energy distribution, which leads to very high recoil energies in 3b and 4b encounters. Also the number of escaping binaries, in our Fig. 13 matches well the HA92 results again (their Fig. 14).

The central escape speed as a function of time, which is a direct measure of the central potential, is presented in Fig. 14, as compared to Fig. 15 of HA92. The N-body models of HA92 reach significantly deeper values of the central potential. This is partially related to the fact that the 5% Lagrangian radius in HA92 is smaller than in our simulations. We think that the reason for that, at least partly, is a higher energy generation in our models, manifesting itself through a higher binary destruction rate (see discussion of Fig. 15) and a bias originating from the pairwise potentials of the single stars. Hence the potential computed in the direct N-body model is more prone to clumps of particles, which we do not have in our gaseous model representation. Also the maximum is at a different time, in our model at the time of maximum binary segregation (roughly at $t \approx 200$) to the centre, in their model at the time of maximum collapse of the single stars ($t \approx 500$).

In Fig. 15 we provide a snapshot of the binary distribution in energy-radius space of the cluster, for a time late well in the escaper phase, for model S. In the late phases a bimodal binary distribution is observed, soft and hard binaries deep in the core, and a group of hard and very hard binaries being ejected into the very outer regions of the cluster. Very similar behaviour can be seen in Fig. 22 of HA92, which gives the corresponding N-body data.

Fig. 16 shows the distribution of binding energies of the binaries as a function of time for model S (see Fig. 16 of HA92); the results again agree rather well, except for the bins of the most bound binaries (between 32 and 128), which are more abundant in our model, due to the relatively early creation of new three-body binaries in the core. The time at which such first creation of new binaries occurs can vary strongly as a function of the random number initialization of the model. Note that the units of binary binding energy are different by an approximate factor of 2 between HA92 and our models; corresponding lines can be identified from $t = 0$.

To complete our survey of results we present the fraction of binaries in the core (Fig. 17) and the core radius (Fig. 18) from the four models S, D, E, and SS (see in HA92 Figs. 17 and 18). A quantitative comparison though is difficult due to very noisy data of N-body and stochastic Monte Carlo models and due to the different and incompatible core definitions in both models. However, the initial increase of the binary fraction in the core, and subsequent slow decrease, are clearly visible in Fig. 17. In Fig. 18 also the initial fast decrease of the core radii for all models is clearly seen, but the further evolution, particularly for model D is not fully compatible with results of HA92 (see their Fig. 18). Note

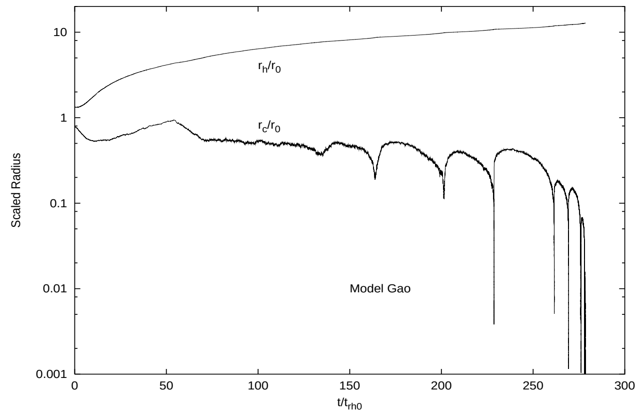

Figure 19. Evolution of the core and half-mass radii (scaled by the scale length of the Plummer model) as a function of time in units of initial half-mass relaxation time; this time unit is used in all following figures.

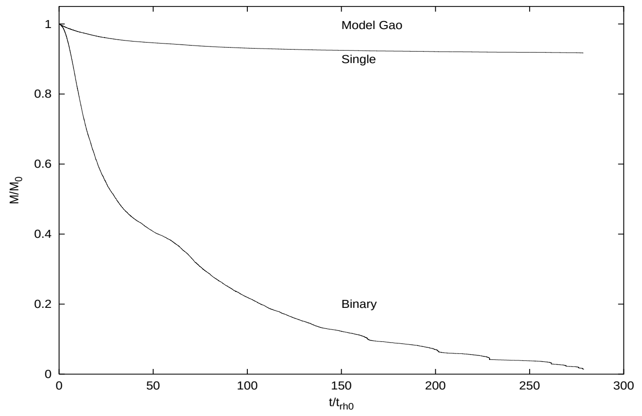

Figure 20. Mass fraction remaining in single stars and binaries as a function of time.

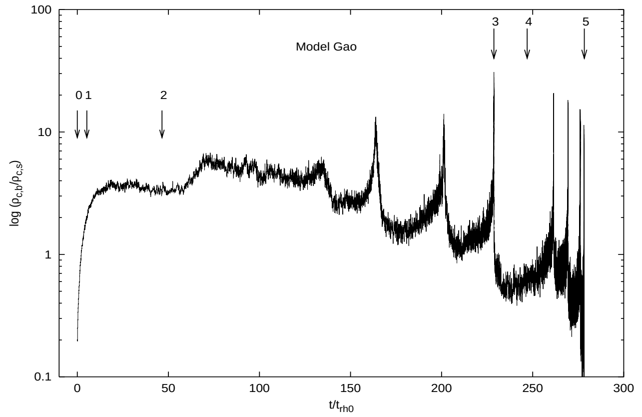

Figure 21. Ratio of central densities in binaries and singles as a function of time. Arrows indicate the times for snapshot plots of 2D and 3D distributions of binaries bound to the system. The selection of times is explained in the text.

that the above mentioned features, which are in good agreement, are related to the initial fast mass segregation phase of the binaries.

3.4 Gao's Runs

Gao's runs are named after the pioneering paper of Gao et al. (1991, GGCM91), which we use as a reference paper. This is the only paper providing models of live and self-consistent single and binary stars in a large number (30.000 primordial binaries) in a star cluster consisting of 360.000 stars (including the binary members). Their models were two-component Fokker-Planck models, treating the binaries as a smooth evolving mass distribution (since all stars have the same mass, only two-components are sufficient). The initial primordial binary distribution has a binding energy range from 3 kT to 400 kT, their spatial distribution is a Plummer model realization using the same scaling radius as for the single stars. In this subsection we describe, how we redo their calculation with our stochastic Monte-Carlo model, what features can be reproduced, and which ones are different. We include the same physics as in GGCM91 as far as possible and reasonable. This means for example we use exactly the same 4b encounter cross-sections that they did. However, our model provides more details of the binary evolution and of the close encounters. For example, about the single star reaction products, which they could not provide due to the statistical treatment of their binaries.

There is one technical remark to be made, regarding the very important choice of a proper relaxation interval for the Monte Carlo binaries (compare standard description in Paper I). In contrast to Heggie's runs, here phases occur where there are very high densities of single stars and binaries. Binaries may stay in very elongated orbits, so if we pick binaries only at some fraction of their orbital time they are being "left behind" in an unphysical way in collapse phases of the single stars. In such a situation the results did not agree with other models or physical expectations. For those binaries newly obtained positions were very far outside, and practically no evolution of the binary population took place. Therefore we employ (after some experiments) the prescription to use the local relaxation time at two times the core radius, divided by ten, as the proper interval for relaxation. It would not be harmful to use an even smaller interval, but then more and more computational time would be spent to relax the binaries. A too small relaxation interval, however, would get into conflict with the Monte Carlo method's principles, because the picking of the binaries position at any point of the orbit becomes unphysical if done in a time interval much smaller than the orbital time scale.

Fig. 19 shows the time evolution of the scaled core and half-mass radii of the single stars, to be compared with Fig. 1 of GGCM91. Like in their model the evolution can be divided into a first, rather fast, mass segregation phase of the binaries (as in the case of Heggie's models), a binary burning phase, in which gradually most of the binaries are destroyed or ejected by close 3b and 4b encounters, and finally a standard phase of gravothermal oscillations follows, which is dominated by the single stars only with the formation of a few new binaries due to 3b encounters. In this phase there are only a few hundred primordial binaries left in the outskirts of the cluster, as will become clear later. The model continues until about 280 initial half-mass relaxation times ($t_{\rm rh0}$). To assess what a heroic effort such a model would

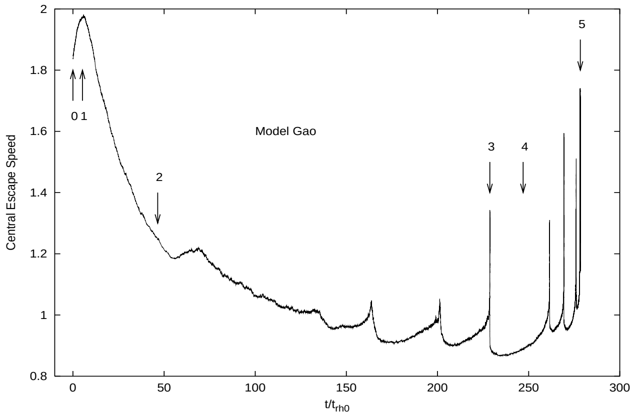

Figure 22. Central escape speed as a function of time. Arrows as explained in the Fig. 21.

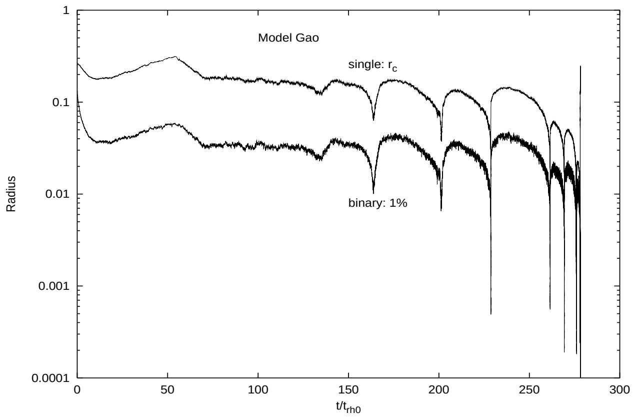

Figure 23. Evolution of the core radius of single stars and 1% Lagrangian radius of binaries as a function of time.

be in the direct N-body approach just note, that this corresponds to 0.8 million N-body time units or some 260.000 initial half-mass crossing times. We stop the model at about 280 $t_{\rm rh0}$, because nearly all binaries have been destroyed or ejected from the core, and the simulation goes on as a single star gravothermally oscillating model, where only binaries are formed in the high-density phases in the core, as in the Monte-Carlo runs discussed in subsection 3.2. Different to those models, we only have a certain cloud of a few hundred binaries in parking orbits very far in the halo, but they do not influence the core evolution. Gao's runs reached a similar state already after 90 $t_{\rm rh0}$; but note that the fraction of time spent in the different phases (such as oscillation or binary burning) seems at a first glance not much different from Gao's case. We will come back to this time scale problem below.

First, we would like to refer the reader to the following figures, which show the mass fraction remaining in singles and binaries bound to the system (Fig. 20), the ratio of central densities of binaries over single stars (Fig. 21), the central escape speed as a measure of central potential (Fig. 22), and the evolution of the core radius of single stars and the 1% Lagrangian radius of the binaries (Fig. 23), all as a function of time in units of $t_{\rm rh0}$. Corresponding figures in GGCM91 are Figs. 2 (mass fractions), and 3 (ratio of central densities). We will now deduce the scenario of the evolution of our system from these figures, with additional information from pictures of the detailed binary distribution in energy (over initial kT) and radius (over initial core radius), provided for key time points in Figs. 24 and 25. The time points are marked by arrows with numbers 0 to 5 in Figs. 21 and 22, corresponding times in $t_{\rm rh0}$ are given in the caption of Fig. 24. We will denote the binary snapshots at these times in short with snapshot number 0 to 5; they show the frequency of binaries (as a 3D surface, with contour levels projected onto the plane below) over a logarithmically equally spaced mesh in scaled energy and position (Fig. 24), or a direct projection of these data (one cross per individual binary) in Fig. 25. We deduce the following basic scenario:

First, up to about 10 $t_{\rm rh0}$ we have a fast binary mass segregation phase, in which the total number of binaries does not change much, since the time scale for their destruction and ejection is still too long, but as the binary distribution flattens out into core and halo, the maximum of the distribution becomes shallower. The effect can be clearly seen in the difference between snapshot 0 and 1, and by the pronounced maximum of central escape speed reached at time point 1 in Fig. 22. Subsequently there follows a phase of rapid binary destruction, as can be seen in Fig. 20 up to about 30 $t_{\rm rh0}$; remarkably from the snapshot 2 it can be seen that preferrably the low energy binaries (up to a few tens of kT) are being destroyed, while the high energy binaries are not affected much. Here we see the 4b encounters being effective: with a high probability the low energy binaries are undergoing a close encounter with a moderate or high energy binary, leading to the destruction of the first and hardening of the second binary. Depending on the energy of the second binary, escape of single stars only or of single stars and the hard binary may happen; if in the beginning there are still enough binaries of moderate binding energy the dominant effect is just heating of the surrounding core by 3b and 4b encounters. Therefore we observe a quasi-stationary binary burning phase lasting up to about 60 $t_{\rm rh0}$, during which the ratio of binary and single star density is kept nearly constant in the core (Fig. 21), with a gradually expanding core radius of the single stars (Fig. 19). At about 60 $t_{\rm rh0}$ the reservoir of binaries to destroy and heat the core is substantially depleted, the binaries have been reduced in number by 60%.

The core starts its first attempt to collapse gravothermally, which is visible as a transient recollapse of the core radius, a shoulder in the binary mass fraction (dM/dt of the binaries becomes significantly smaller temporarily), and an increase of the ratio of binary to single star density, as a consequence of mass segregation, all three effects clearly visible between 50 and 70 $t_{\rm rh0}$ in Figs. 19 to 23. The onset of the first attempt to gravothermal collapse is rather abrupt. As binary burning becomes too weak the core starts to collapse due to a smaller proportion of binaries, but this is stopped here after more binaries have again segregated to the centre. Thus, there are still enough binaries (a few tens of thousands) to halt core collapse at a higher level of binary to single star density, and cause a second quasi-stationary phase until about 150 $t_{\rm rh0}$. The process repeats itself, but now at a small enough binary number that the single stars can start to undergo the first pronounced oscillatory peaks (small core radius). Gravothermal oscillations follow, which

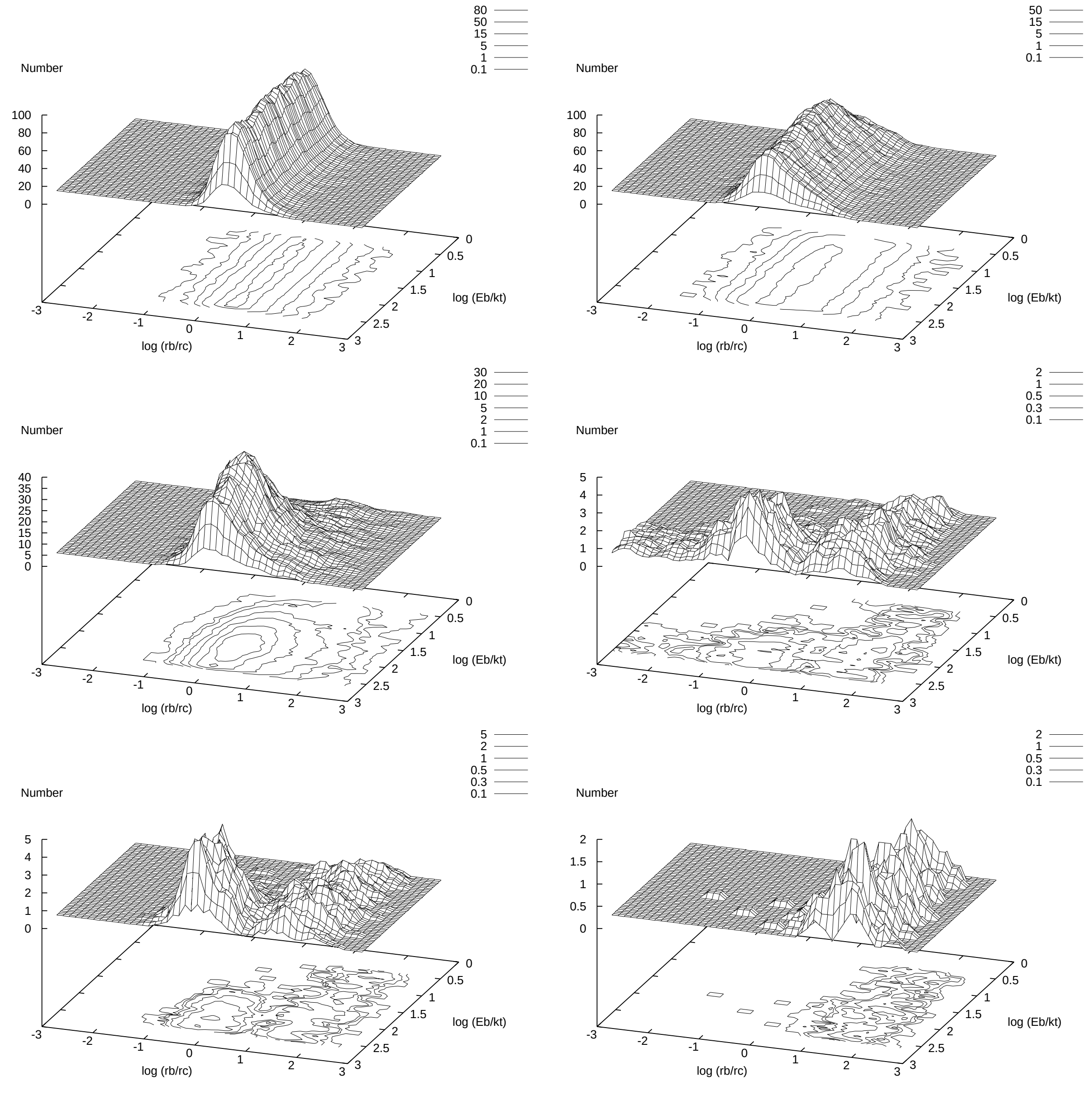

Figure 24. Snapshot plots of 3D distributions of bound binaries. Plotted is the frequency of binaries in logarithmically equidistant bins of binding energy in kT (initial) and position in units of initial core radius, for six different times in units of the initial half-mass relaxation time; top left 0, top right 5.21, middle left 46.51, middle right 228.70, bottom left 246.99, bottom right 278.36. The selection of times is the same as indicated by the arrows in Fig. 22.

are pronounced at the time points 3 (maximum density), 4 (expanded stage) and 5 (terminal point of our model). It is interesting to note, that the gravothermal oscillations are not only visible in the core radius and Lagrangian radii of the single stars. They are also visible in the binary density, and in the *ratio* of binary to single star density (see Figs. 21 and 23). These features, including amplitude variations over many orders of magnitude, multi-peak structure with period doublings at the maxima, and long inactive expanded phases are clear and the same as observed in GGCM91 and other standard models of gravothermal oscillations. From Fig. 24, time points 3 and 4 we can see, that in the collapsed phase (time point 3) some binaries are very deep in the core, while in the expanded phase (time point 4) the core is void of binaries, no activity taking place. Most interestingly at the final time point 5, we have reached a situation where all binaries are very far outside in parking orbits. We note, however, that from the studies of MC runs (MC5 and MC5Q, subsection 3.2) the reason, why there are so many binaries in parking orbits is not yet clear. In contrast to MC runs, however, here most binaries in parking orbits are still primordial and have not been created in 3b encounters in high density phases. Therefore we believe that the existence of many binaries in such orbits is real, though our model may overestimate it.

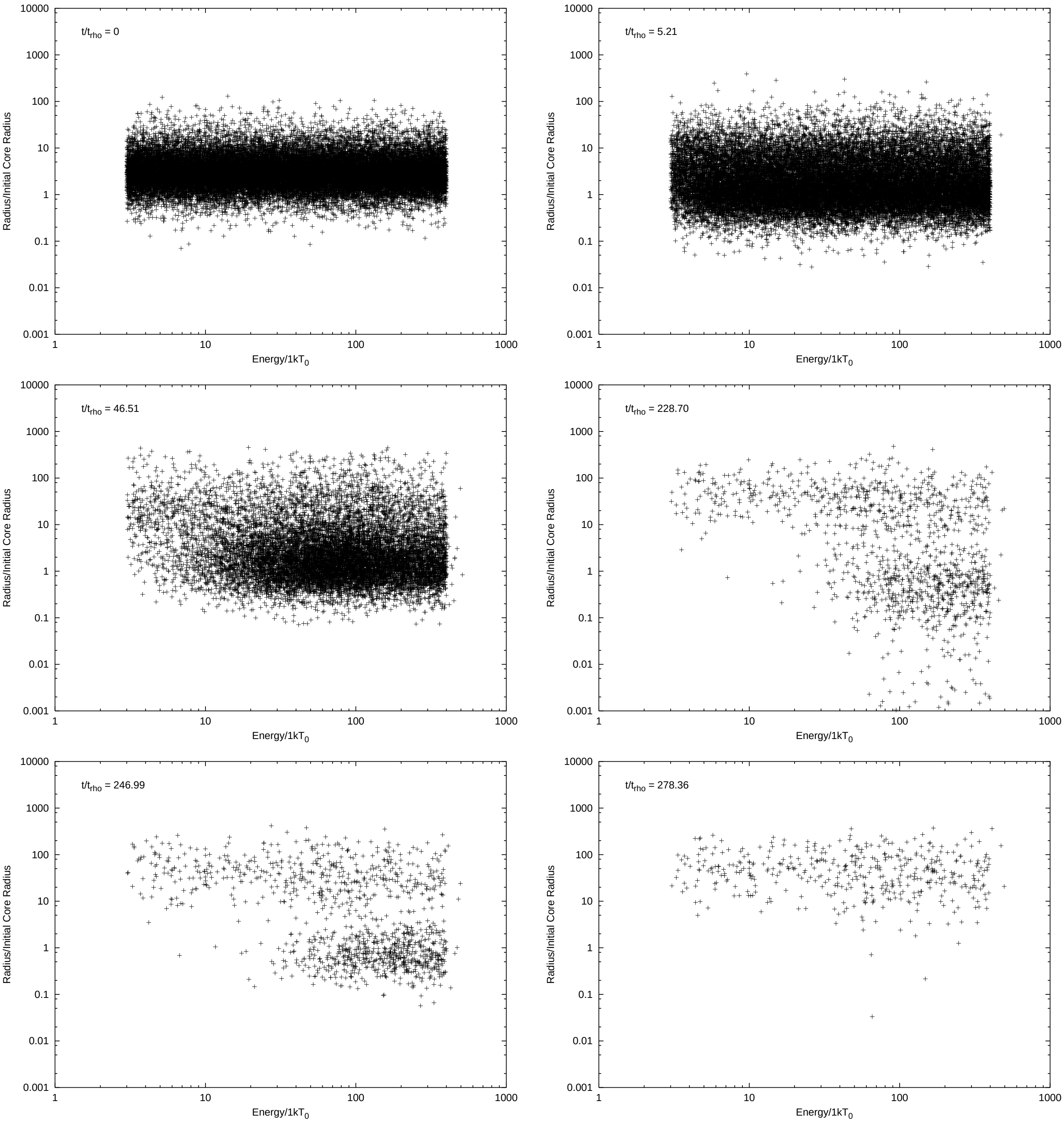

Figure 25. The same data as in Fig. 24, but in a 2D projection of the individual data of each binary (being represented by a cross) onto the energy-radius plane, all units as in Fig. 24.

In any case, such binaries decouple from the evolution of the rest of the system, which will behave more like a single star cluster with gravothermal oscillations. Binaries left over in the outer halo may be regarded as fossil records of the early primordial binary generation. They will be slowly depleted by 3b and 4b interactions when they enter the core due to relaxation processes. Note that in phases 3 to 5 the scale of the binary distribution in Fig. 24 has been amplified by a large factor; the absolute frequency (and total number) of binaries are one to two orders of magnitude smaller than at time points 0 to 2, as can be deduced directly from Fig. 20.

Figs. 26, 27, and 28 show the evolution of selected Lagrangian radii for binaries and singles, the number of escaping stars in binaries, singles, and total (counting binaries as two stars), and the total energy budget, in the same way as discussed for Heggie's models. Due to their stronger central concentration the binaries take part in the gravothermal oscillations with more than 50% of their total mass; at the end the final complete removal of binaries from the core can be seen in Fig. 26. Figs. 27 and 28 show that the initial binary destruction phase is accompanied also by a heavy loss of binaries with high binding energy, as can be seen from the strong drop of $E^{e}_{\rm int}$. In the following phase the rate of escape becomes slower, but the mechanism is the same, energy bound in binaries is carried away by escapers. Note the signature of the oscillations in the binary escaper energy at late times, and the enormous amount of energy exchanged via the binaries, as compared to the initial and final total external energy of the star cluster, which starts at the standard value of -0.25. Remarkably we lost in total at the end of our simulation about 90.000 stars escaped altogether, which is one quarter of the initial number of stars. This number would be even larger if escapers due to relaxation effects of single stars were allowed. Since we lose only some 15.000 binaries by escape (Fig. 27), it can be concluded that another 15.000 binaries have been destroyed by close 4b encounters, and, more interestingly, at least 30.000 single stars have escaped which do *not* originate from one of the primordial binaries, assuming that the maximum of all destroyed binaries led to two single escapers, which is an upper limit only, and neglecting possible exchange reactions. In other words, per binary destroyed or ejected about three single stars escaped on average. Looking at Fig. 2 of GGCM91 they find an increased number of single stars during the first evolutionary phase, which cannot be seen in our results. We think that this difference is an artifact of the GGCM91 runs, due to their rather artificial procedure to select binaries from energy bins for close encounters, while in our paper the proper encounter probabilities are used. Thereby we destroy less quickly the soft binaries and find a higher fraction of escaping single stars from the beginning, so in our results the number of bound single stars decreases. Another reason why GGCM91 find more bound single stars is that they can not take into account the external energy of the destroyed binary in a 4b encounter, as a further contribution to the possibly escaping single stars in excess of the recoil energy obtained by hardening the harder of the two binaries (though the latter will in most cases be the dominant contribution).

Hence, we can explain the time scale differences between our and GGCM91 models; we keep a larger number of soft or intermediate binaries in the system which are able to supply sufficient binary heating, by single stars originating from 4b encounters and remaining in the core. These heating reactions support our long initial binary burning phase, which takes place in two phases (as discussed above) until 150 $t_{\rm rh0}$. Opposed to this, the soft and intermediate binaries in GGCM91 are quickly destroyed, leading to a much stronger collapse of the single star and the remaining relatively hard binaries, and further acceleration of the recoil and escape process. Consistently one could see that the gravothermal oscillations in GGCM91 already begin at a fraction of binaries left of about 30 to 40%, while in our case we have a binary depletion to 10% until the oscillations start. Since they destroy the weak binaries too quickly this difference in evolution between the models occurs, although the time evolution of the total binary number is not so different. All this leads to a time, to begin gravothermal oscillations, which is about a factor of 3 shorter than in our models. This is a big difference, which however, can be fully explained by a much faster binary destruction in their model. We attribute this to the rather artificial procedure by which GGCM91 decide whether a close 4b encounter takes place.

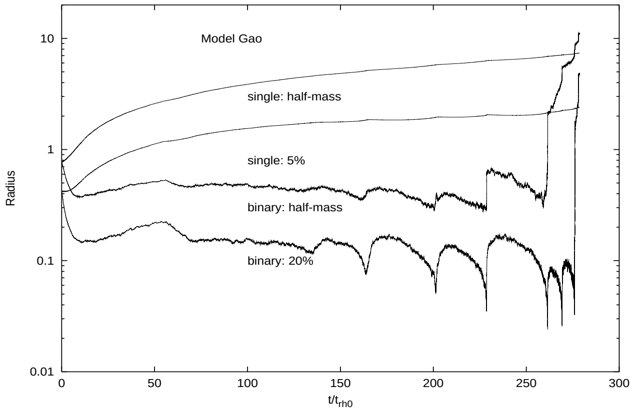

Figure 26. Evolution of Lagrangian radii containing 50% and 5% of the mass of single stars and 50% and 20% of the mass of binaries as a function of time.

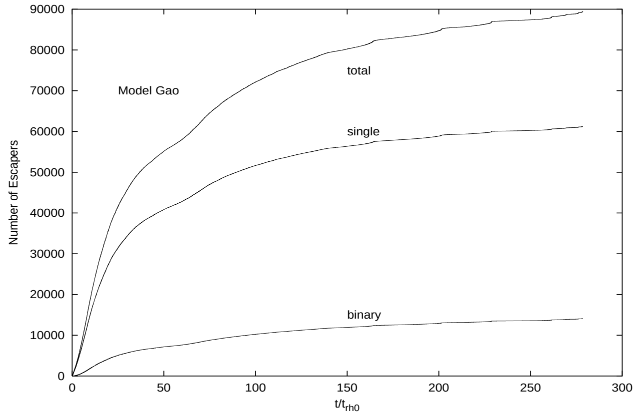

Figure 27. Number of escaping single stars, binaries and the total number of escapers as a function of time.

4 CONCLUSIONS AND DISCUSSION

The new approach outlined in Paper I (Spurzem & Giersz 1996) to follow, in the Monte Carlo manner, the individual formation and evolution of binaries in an evolving point-mass cluster was successfully extended to the fully self-consistent treatment of relaxation and close three- and four-body (abbreviated 3b and 4b) encounters for a substantial number (typically a few percent of the initial number of stars) of binaries and a realistic total number of stars. We use a standard anisotropic gaseous model (Louis & Spurzem 1991, Spurzem 1994), describing the single stars, and the Monte Carlo technique (Giersz 1998) to model, in a stochastic way, the binary subsystem. The aim of this paper was to test the reliability of the new hybrid code by comparing its results with the data available in the literature for single mass systems (full Monte Carlo model - Giersz 1998), and for systems with substantial primordial binary population (N-body models - HA92 and Fokker-Planck models -

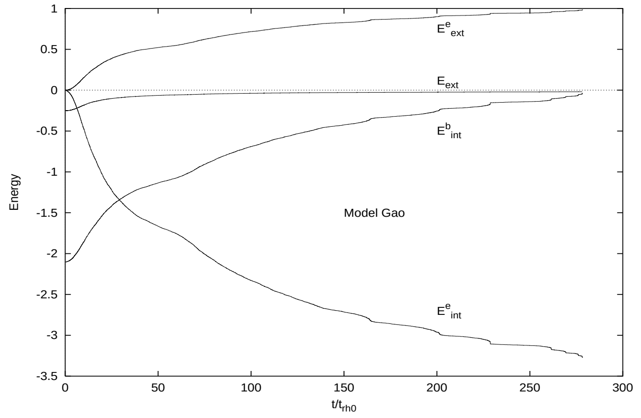

Figure 28. Energy balance as a function of time. The four different contributions are described in the text, and are the same as used for Fig. 10 for Heggie's runs.

GGCM91). Also we want to show, that our model is able to handle large N systems with a large binary number by a reasonable effort, but keeps full self-consistency and detailed information. It will be possible in the near future, after some more real physics is included (such as stellar evolution, finite stellar radii, tidal fields, and a mass spectrum), to provide unique model data for comparisons with the expected wealth of observational data for globular clusters.

The features of our MC5 and MC5Q runs for isolated, single mass system consisting of 10^5 stars with only strong 3b (MC5) and both 3b and 4b (MC5Q) interactions, respectively, are in a good agreement with the full Monte Carlo model (Giersz 1998). The gravothermal oscillations are the most pronounced features of these runs. They show very large amplitude in central density, long expanded (inactive) phases, and several discrete oscillation frequencies which are observed in standard gaseous or Fokker-Planck models of gravothermal oscillations (Bettwieser & Sugimoto 1984, Heggie & Ramamani 1989, Cohn, Hut & Wise 1989, Breeden et al. 1994). Also rapid changes in the phase of the oscillations occur, which agree with other stochastic Fokker-Planck, N-body and Monte Carlo models (Takahashi & Inagaki 1991, Makino 1996 and Giersz 1998). As can be expected, the expansion phases of the system are powered by the themperature inversion in the core, not by the energy generated by binaries in interaction with singles and other binaries. In the large expansion phases there is practically no binary activity, as expected. The binary distribution in energy-radius space of the cluster is clearly bimodal. Binaries with high binding energies and in orbits not entering the core and extended far into the halo form one group. Binaries with a wide range of binding energies and orbits entering the core form the second group. From models with artificially suppressed 4b encounters we find that they cause a wider distribution of binary binding energies (mainly towards larger binding energies) and a somewhat less pronounced bimodal binary distribution in a diagram showing the binary distribution in radius and binding energy. Our models exhibit a rather large number of binaries in parking orbits (orbits which do not reach the core, and have apocentre very far out in the halo), as compared to full Monte Carlo (Giersz 1998) and direct N-body models (Spurzem & Aarseth 1996). The reason is unclear, but see a discussion in subsections 3.2 and 3.3. In any system with even a slight external tidal field such binaries will be removed very fast.

For runs with primordial binaries we discuss three different cases.

First, we use an artifical model with two-components, one consisting of single, the other of binary stars, which only interact by relaxation with each other and with the single (all close 3b and 4b encounters were suppressed). We check the rate of mass segregation and core collapse time by comparison with a continous two-component anisotropic gaseous model (Louis & Spurzem 1991, Spurzem 1994), just in the same way as HA92 checked their N-body models by comparison with one of Heggie's gaseous models. We find that our stochastic Monte Carlo model of the binary component provides a good match with the expectation from the gaseous model. While in Paper I we only checked the dynamical friction of one binary in a system of single stars, this is a more significant test of the self-consistent treatment of relaxation in our model.

Second, a set of four models corresponding to the largest published full N-body models with primordial binaries by Heggie & Aarseth (1992, HA92) was performed. We find a good agreement of our stochastic Monte Carlo model for most features, as they were published in HA92. The remaining differences regarding the binary number evolution, central potential, and fraction of hard binaries in the system between their and our models can be related to stochastic variations, as they usually occur in individual Monte Carlo realizations of the system.

Third, and as a final goal of this paper, we have performed a self-consistent model of 30.000 binaries evolving internally and externally, surrounded by a live star cluster of 300.000 single stars, undergoing mass segregation, core collapse, and finally large amplitude gravothermal oscillations. The binaries during all these evolutionary phases are subject to binary destructions and ejections by close 3b and 4b encounters. This model is called Gao's run, to remind the reader of the pioneering work of Gao et al. (1991, GGCM91), which performed for the first time a self-consistent two-component Fokker-Planck model of binaries and single stars. Our aim was, in this paper, to redo this model as much as possible and reasonable, keeping most of the assumptions, such as the 3b and 4b interaction cross sections, the assumption of single mass (all binaries consist of mass components equal to the single stars), and the assumption of an isolated point-mass system. However, in addition to their model, we exploit in this framework the strengths of our stochastic Monte Carlo model, such that we are able to follow the individual evolution of internal and external parameters of any binary in nearly the same detail as in an N-body model, provide snapshots of the position and binding energy of all the binaries at crucial times of the evolution (first mass segregation, binary heating and destruction phase, gravothermal oscillation phase), which can be seen by the interested reader as a movie under

`ftp://ftp.ari.uni-heidelberg.de/pub/spurzem/movie?.mpg`

where "?" refers to 1 or 2, depending whether one wants to see the movie in the style of Fig. 24 or 25. Also we do not use the rather artificial procedure of picking binaries in the energy bins for close encounters as used by GGCM91, because in their model no better procedure could be found. Since we know positions of binaries at any time, we can select proper probabilities for close encounters of the 3b and 4b kind to occur and find a much slower destruction rate of soft and intermediate binaries than GGCM91. As a consequence the quasistationary binary burning phase extends over a time in our model (up to 150 $t_{\rm rh0}$) which is about three times larger than the corresponding time in GGCM91. After that, gravothermal oscillations start, and at about 280 $t_{\rm rh0}$ all primordial binaries left the core and its vicinity. Only a fossil collection of some 400 binaries can be found in the very outer halo, as a final trace of the initial primordial binary population. During the maximum density phase new binaries start to be formed by 3b interactions. Of the initially 30.000 binaries, 15.000 were ejected by 3b and 4b encounters, and another 15.000 destroyed by 4b encounters. The single stars originating from such binary destructions are partly heating the cluster, partly escaping, depending on the binding energy of the harder binary in such encounters. On average we find that for any binary, whether destroyed or ejected, about three single stars are ejected. Qualitatively we find the same phases as GGCM91, but our evolution time scales are a factor of 3 longer. Though the evolutionary time scale is much longer than the Hubble time, we think our models are still (already on the present idealized level) able to provide some insight into the physical evolution of large star clusters with many primordial binaries, at least if accepting the point mass approximation. Note, that the inclusion of more realism (see below) according to previous experiences always shortens the evolutionary lifetimes of clusters, and stellar mass loss will change the cluster's and the individual binaries evolution dramatically. Nevertheless we consider our work as an important and indispensable step towards such models with very large binary and single star numbers.

Admittedly our model is still not realistic for a globular cluster in many respects. The evolution of hard binaries during close encounters is known to be *very* different from that of equal mass stars; stellar evolution and finite size effects will dramatically alter the channels through which primordial binaries are processed (see Hut, McMillan & Romani 1992). Tidal fields may strip the outer areas of the cluster, especially those where we find our remaining binaries at the end. Nevertheless we stress, that in our model we have achieved a breakthrough in the homogeneous, self-consistent treatment of very many binaries in a large star cluster. We model the cluster evolution during 0.8 million N-body time units, which are some 260.000 initial half-mass crossing times, without either using special hardware or other than the fundamental assumptions of spherical symmetry and dominance of small angle two-body encounters (which are very robust assumptions in large N systems). Such a job is enormous, even for the next generation of Petaflop computers if done by a direct N-body model. In this sense our model has proven its usefulness, uniqueness, and pioneering capabilities, which gives credit to those people, who have been developing the Monte-Carlo method (Hénon 1971, Spitzer 1975, Stodółkiewicz 1982, 1986), and it shows the relevance of the new Monte Carlo models developed recently (Giersz 1996, 1998).

The yet missing effects (multi-mass, stellar evolution, tidal fields, more detailed cross-sections, obtained by direct modelling) will be included into our models for the near future, and do not pose a fundamental challenge. But in order to do the proper astrophysical study, we rather prefer to do it step by step, connecting our models with Heggie's, Giersz's and Gao's runs, discuss their reliability, agreement and disagreement and the physical reasons of it, to understand what goes on in our model, as compared to just start with one big model containing everything.

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REFERENCES

Aarseth S.J., 1985, in Brackbill J.U., Cohen B.I., eds, Multiple time scales, Academic Press, Orlando, p. 378
Aarseth S.J., 1996, in Hut P., Makino J., eds, Proc. IAU Symp. 174, Dynamics of Star Clusters, Reidel, Dordrecht, p. 161
Aarseth S.J., 1999, CeMDA, 73, 127.
Aarseth S.J., Heggie D.C., 1993, in Smith G.H., Brodie J.P., eds, ASP Conf. Ser. 48, The globular clusters-galaxy connection, p. 701
Aarseth S.J., Heggie D.C., 1998, MNRAS, 297, 794
Bettwieser E., Sugimoto D., 1984 MNRAS, 208, 493
Breeden J.L., Cohn H.N., Hut P., 1994, ApJ, 421, 195
Cohn H., 1980, ApJ, 242, 765
Cohn H., Hut P., Wise M.W., 1989, ApJ, 342, 814
Casertano S., Hut P., 1985, ApJ, 298, 80
Einsel Ch., Spurzem R., 1999, MNRAS, 302, 81
Gao B., Goodman J., Cohn H., Murphy B., 1991, ApJ, 370, 567, GGCM91
Giersz M., 1996, in Hut P., Makino J., eds, Proc. IAU Symp. 174, Dynamics of Star Clusters, Reidel, Dordrecht, p. 101
Giersz M., 1998, MNRAS, 298, 1239
Giersz M., Heggie D.C., 1994a, MNRAS, 268, 257, GHI
Giersz M., Heggie D.C., 1994b, MNRAS, 270, 298, GHII
Giersz M., Heggie D.C., 1997, MNRAS, 286, 709
Giersz M., Spurzem R., 1994, MNRAS, 269, 241, GS
Heggie D.C., 1984, MNRAS, 206, 179
Heggie D.C., Aarseth S.J., 1992, MNRAS, 257, 513, HA92
Heggie D.C., Giersz M., Spurzem R., Takahashi K., 1998, in Andersen J. (ed), Highlights of Astronomy Vol. 11. Kluwer, Dordrecht. (Preprint astro-ph/9711197)
Heggie D.C., Mathieu R.M. 1986, in Hut P., McMillan S.L.W., eds, The Use of Supercomputers in Stellar Dynamics. Springer-Verlag, Berlin, p. 233
Heggie D.C., Ramamani N., 1989, MNRAS, 237, 757
Hénon M., 1971, Ap&SS, 14, 151
Hénon M., 1975, in A. Hayli, ed, Proc. IAU Symp. 69, Dynamics of Stellar Systems, Reidel, Dordrecht, p. 133
Hut P., et al., 1992, PASP, 104, 981
Hut P., McMillan S., Romani R.W., 1992, ApJ, 389, 527
Inagaki S., Hut P., 1988, in Valtonen M.J., ed, Proc. IAU Coll. 96, The Few Body Problem, Kluwer, Dordrecht, p. 319
Joshi K.,J., Nave C. & Rasio F.A., 2000, ApJ submitted (astro-ph/9912155)

Joshi K.,J., Rasio F.A. & Portegies Zwart S., 2000, ApJ submitted (astro-ph/9909115)
Kroupa P., 1995, MNRAS, 277, 1491
Lee H.M., Fahlman G.G., Richer H.B., 1991, ApJ, 366, 455
Louis P.D, Spurzem R., 1991, MNRAS, 251, 408
Makino J., 1996, in Hut P., Makino J., eds, Proc. IAU Symp. 174, Dynamics of Star Clusters, Reidel, Dordrecht, p. 141
Makino J., Aarseth S.J., 1992, PASJ, 44, 141
Makino J., Taiji M., 1998, Scientific simulations with special purpose computers, Chichester, Wiley
Makino J., Taiji M., Ebisuzaki T., Sugimoto D., 1997, ApJ, 480, 432
McMillan S.L.W., Engle E.A., 1996, in Hut P., Makino J., eds, Proc. IAU Symp. 174, Dynamics of Star Clusters, Reidel, Dordrecht, p. 379
McMillan S.L.W., Hut P., Makino J., 1990, ApJ, 362, 522
McMillan S.L.W., Hut P., Makino J., 1991, ApJ, 372, 111
McMillan S.L.W., Hut P., 1994, ApJ, 427, 793
McMillan S.L.W., Hut P., 1996, ApJ, 467, 348
Mikkola S., 1983a, MNRAS, 203, 1107
Mikkola S., 1983b, MNRAS, 205, 733
Mikkola S., 1984a, MNRAS, 207, 115
Mikkola S., 1984b, MNRAS, 208, 75
Portegies Zwart S.F., Hut P., Makino J., McMillan S.L.W., 1998, A&A, 337, 363
Sigurdsson S., Phinney E.S., 1995, ApJS, 99, 609
Spitzer L., 1975, in Hayli A., ed, Dynamics of Stellar Systems, Reidel: Dordrecht, p.3
Spitzer L., 1987, Dynamical Evolution of Globular Clusters, Princeton Univ. Press, Princeton
Spurzem R., 1992, in Klare G., ed, Reviews of Modern Astronomy 5, Springer Verlag Berlin Heidelberg, p.161
Spurzem R., 1994, in Pfenniger D., Gurzadyan V.G., eds, Ergodic Concepts in Stellar Dynamics, Springer-Vlg., Berlin, Heidelberg, p. 170
Spurzem R., 1996, in Hut P., Makino J., eds, Dynamics of Star Clusters, Proc. IAU Symp. No. 174, p.111
Spurzem R., 1999, in Riffert H., Werner K. (eds), Computational Astrophysics, The Journal of Computational and Applied Mathematics (JCAM) 109, Elsevier Press, Amsterdam, p. 407
Spurzem R., Aarseth S.J., 1996, MNRAS, 282, 19
Spurzem R., Giersz M., 1996, MNRAS, 283, 805, Paper I
Spurzem R., Takahashi K., 1995, MNRAS, 272, 772
Stodółkiewicz J.S., 1982, Acta Astronomica, 32, 63
Stodółkiewicz J.S., 1985, in Goodman J., Hut P., eds, Proc. IAU Symp. 113, Dynamics of Star Clusters, Reidel, Dordrecht, p. 361
Stodółkiewicz J.S., 1986, Acta Astronomica, 36, 19
Sugimoto D., Chikada Y., Makino J., Ito T., Ebisuzaki T., Umemura M., 1990, Nature, 345, 33
Takahashi K., 1995, PASJ, 47, 561
Takahashi K., 1996, PASJ, 48, 691
Takahashi K., 1997, PASJ, 49, 547
Takahashi K., Inagaki S., 1991, PASJ, 43, 589
Takahashi K., Lee H.M., Inagaki S., 1997, MNRAS, 292, 331